Problem 1

a) If we are given $X_k$, we know that there are $26 - X_k$ red cards and $26 - (k - X_k)$ black cards. Since the probability of guessing right (or wrong) when next card is drawn depends only on the number of red and black cards left, given $X_k$, it is independent of the past $(X_{k-1}, X_{k-2}, ...)$). Therefore, $\{X_k\}$ is a Markov chain.

The probability of guessing right (or wrong) when next card is drawn depends on $k$ (because the number of black cards left is $26 - (k - X_k)$ and obviously depends on not only $X_k$ but also $k$).

Therefore, $\{X_k\}$ is not homogeneous.

b) $P[X_7 = 6|X_6 = 5] = \text{Prob. that the player guesses right when there were 21 reds and 25 blacks before the drawing of the 7th card}$

\[ = \frac{21}{46} \]

c) $P[X_9 = 4|X_8 = 2] = \text{Prob. that the player gets 2 dollars more after the drawing of the 9th card}$

\[ = 0 \]

d) Given $X_k = i$, we know that there are $(26 - i)$ reds and $(26 - (k - i))$ blacks. Therefore, when $j = i + 1$:

$P[X_{k+1} = j|X_k = i] = \frac{26 - i}{52 - k}$

and when $j = i$:

$P[X_{k+1} = j|X_k = i] = \frac{26 - (k - i)}{52 - k}$

For all other values of $j$:

$P[X_{k+1} = j|X_k = i] = 0$
Problem 2

a) The population of the \((k+1)\)th generation is

\[ X_{k+1} = Y_1 + Y_2 + \ldots + Y_{X_k} \]

We have that

\[ P[X_{k+1} = x_{k+1} | X_k = x_k] = P[X_{k+1} = x_{k+1} | X_k] \]

\[ = P[Y_1 + Y_2 + \ldots + Y_{X_k} = x_{k+1}] \]

Hence, \( \{X_k\} \) is a Markov chain. It is also homogeneous because the transition probabilities \( P[X_{k+1} = x_{k+1} | X_k = x_k] \) do not depend on \( k \).

b)  

\[ \begin{array}{c}
0 \quad 1/3 \quad 1/3 \quad 1/3 \\
1/3 \quad 0 \quad 2/3 \quad 1/3 \\
2/3 \quad 1/3 \quad 0 \quad 2/3 \\
1/3 \quad 1/3 \quad 2/3 \quad 0 \\
2/3 \quad 1/3 \quad 1/3 \quad 2/3 \\
1/3 \quad 2/3 \quad 1/3 \quad 0 \\
\end{array} \]

The above is a state transition diagram for the states 0,1,2,3.

The transition probabilities are:

\[ P[X_{k+1} = 0 | X_k = 0] = 1 \]

\[ P[X_{k+1} = j | X_k = 0] = 0 \quad \forall j > 0 \]

\[ P[X_{k+1} = j | X_k = 1] = \begin{cases} 
1/3 & \text{for } j = 0 \\
1/3 & \text{for } j = 1 \\
2/3 & \text{for } j = 2 \\
0 & \text{for } j > 2 
\end{cases} \]

\[ P[X_{k+1} = j | X_k = 2] = \begin{cases} 
1/9 & \text{for } j = 0 \\
1/3 & \text{for } j = 1 \\
1/3 & \text{for } j = 2 \\
2/9 & \text{for } j = 3 \\
1/9 & \text{for } j = 4 
\end{cases} \]
\[
\begin{align*}
\text{c) } P[X_4^* = 0 | X_c = 1] &= \frac{1}{3} \sum_{\text{all } j \neq 0} P[X_4^* = 0 | X_c = j] P[X_j = j | X_c = 1] \\
&= P[X_4^* = 0 | X_c = 1] P[X_2 = 1 | X_c = 1] + P[X_4^* = 0 | X_c = 2] P[X_2 = 2 | X_c = 1] \\
&= P[X_4^* = 0 | X_c = 1] \frac{1}{3} + P[X_4^* = 0 | X_c = 2] \frac{1}{3} \\
&= \frac{1}{3} \sum_{\text{all } j \neq 0} P[X_4^* = 0 | X_c = j] P[X_2 = 1 | X_c = j] + \frac{1}{3} \sum_{\text{all } j \neq 0} P[X_4^* = 0 | X_c = j] P[X_2 = 2 | X_c = j] \\
&= \frac{1}{3} P[X_4^* = 0 | X_c = 1] P[X_2 = 1 | X_c = 1] + \frac{1}{3} P[X_4^* = 0 | X_c = 2] P[X_2 = 2 | X_c = 1] \\
&\quad + \frac{1}{3} P[X_4^* = 0 | X_c = 1] P[X_2 = 2 | X_c = 1] + \frac{1}{3} P[X_4^* = 0 | X_c = 2] P[X_2 = 2 | X_c = 2] \\
&\quad + \frac{1}{3} P[X_4^* = 0 | X_c = 3] P[X_2 = 3 | X_c = 2] + \frac{1}{3} P[X_4^* = 0 | X_c = 4] P[X_2 = 4 | X_c = 2] \\
&= \frac{1}{3} P[X_4^* = 0 | X_c = 1] + \frac{1}{3} P[X_4^* = 0 | X_c = 2] + \frac{1}{3} P[X_4^* = 0 | X_c = 3] + \frac{1}{3} P[X_4^* = 0 | X_c = 4] \\
&\quad + \frac{2}{27} P[X_4^* = 0 | X_c = 1] + \frac{1}{27} P[X_4^* = 0 | X_c = 2] \\
&= \frac{5}{27} P[X_4^* = 0 | X_c = 1] + \frac{2}{27} P[X_4^* = 0 | X_c = 2] + \frac{2}{27} P[X_4^* = 0 | X_c = 3] + \frac{1}{27} P[X_4^* = 0 | X_c = 4]
\end{align*}
\]

\(X^*\) denotes the event: \([X_4 = 0, X_3 \neq 0, X_2 \neq 0, X_1 \neq 0]\)
\[ P[x_4^* = 0|x_2 = 1] = P[x_4^* = 0|x_3 = 1] P[x_2 = 1] + P[x_4^* = 0|x_3 = 2] P[x_2 = 1] + P[x_4^* = 0|x_3 = 3] P[x_2 = 1] \]
\[ = P[x_4^* = 0|x_3 = 1] \frac{1}{3} + P[x_4^* = 0|x_3 = 2] \frac{1}{3} + P[x_4^* = 0|x_3 = 3] \frac{1}{3} = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{4}{27} \quad (i) \]

\[ P[x_4^* = 0|x_2 = 2] = P[x_4^* = 0|x_3 = 1] P[x_2 = 1] + P[x_4^* = 0|x_3 = 2] P[x_2 = 2] + P[x_4^* = 0|x_3 = 3] P[x_2 = 2] \]
\[ + P[x_4^* = 0|x_3 = 3] P[x_2 = 2] + P[x_4^* = 0|x_3 = 4] P[x_2 = 2] \]
\[ = \frac{1}{3} \times \frac{3}{9} + \frac{1}{3} \times \frac{3}{9} + \frac{1}{81} \times \frac{3}{9} + \frac{1}{27} \times \frac{2}{3} + \frac{1}{81} \times \frac{2}{3} + \frac{1}{243} \times \frac{2}{3} = \frac{64}{429} \quad (ii) \]

\[ P[x_4^* = 0|x_2 = 3] = P[x_4^* = 0|x_3 = 1] P[x_2 = 1] + P[x_4^* = 0|x_3 = 2] P[x_2 = 3] + P[x_4^* = 0|x_3 = 3] P[x_2 = 3] + P[x_4^* = 0|x_3 = 4] P[x_2 = 3] \]
\[ + P[x_4^* = 0|x_3 = 5] P[x_2 = 3] \]
\[ = \frac{1}{3} \times \frac{5}{9} + \frac{1}{3} \times \frac{5}{9} + \frac{1}{81} \times \frac{5}{9} + \frac{1}{27} \times \frac{5}{3} + \frac{1}{81} \times \frac{5}{3} + \frac{1}{243} \times \frac{5}{3} = \frac{163}{2187} \quad (iii) \]

\[ P[x_4^* = 0|x_2 = 4] = \sum_{j=1}^{6} P[x_4^* = 0|x_3 = j] P[x_2 = 4] \]
\[ = \frac{1}{3} \times \frac{10}{81} + \frac{1}{9} \times \frac{10}{81} + \frac{1}{81} \times \frac{10}{81} + \frac{1}{27} \times \frac{10}{81} + \frac{1}{243} \times \frac{10}{81} + \frac{1}{81} \times \frac{10}{81} = \frac{22000}{531441} \quad (iv) \]

From (i), (ii), (iii), (iv) and (v) we get

\[ P[x_4^* = 0|x_0 = 1] = \frac{5}{27} \times \frac{1}{27} + \frac{1}{9} \times \frac{61}{243} + \frac{9}{27} \times \frac{163}{2187} + \frac{1}{81} \times \frac{22000}{531441} = 0.053 \]

\[ d) \ P[\text{population remains in existence for more than three generations}] \]
\[ = 1 - P[x_4^* = 0|x_0 = 1] = 0.947 \]
Problem 3

b) 

\[
P^2 = \begin{bmatrix}
0.4 & 0.4 & 0.2 \\
0.5 & 0.3 & 0.2 \\
0.1 & 0.5 & 0.4
\end{bmatrix}
\begin{bmatrix}
0.4 & 0.4 & 0.2 \\
0.5 & 0.3 & 0.2 \\
0.1 & 0.5 & 0.4
\end{bmatrix} = \begin{bmatrix}
0.38 & 0.38 & 0.24 \\
0.37 & 0.39 & 0.24 \\
0.33 & 0.39 & 0.24
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_c(2) & P_1(2) & P_2(2)
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
P^2
\end{bmatrix} = \begin{bmatrix}
0.38 & 0.38 & 0.24 \\
0.37 & 0.39 & 0.24 \\
0.33 & 0.39 & 0.24
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.5 & 0.3 & 0.2
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_c(2) & P_1(2) & P_2(2)
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
P^2
\end{bmatrix} = \begin{bmatrix}
0.37 & 0.39 & 0.24
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
\pi_0 & \pi_1 & \pi_2 \\
\end{bmatrix} = \begin{bmatrix}
\pi_0 & \pi_1 & \pi_2 \\
0.4 & 0.4 & 0.2 \\
0.5 & 0.3 & 0.2 \\
0.4 & 0.5 & 0.4 \\
\end{bmatrix}
\]
\[\pi_0 + \pi_1 + \pi_2 = 1\]
\[\pi_0 = 0.4 \pi_0 + 0.5 \pi_1 + 0.1 \pi_2 \]
\[\Rightarrow \pi_1 = 0.4 \pi_0 + 0.3 \pi_1 + 0.5 \pi_2\]
\[\pi_2 = 0.2 \pi_0 + 0.9 \pi_1 + 0.4 \pi_2\]
\[\pi_0 + \pi_1 + \pi_2 = 1\]

\[0.6 \pi_0 = 0.5 \pi_1 + 0.1 \pi_2\]
\[0.4 \pi_0 = 0.7 \pi_1 - 0.5 \pi_2\]
\[\pi_0 + \pi_1 + \pi_2 = 1\]

\[\pi_0 = \frac{5}{6} \pi_1 + \frac{1}{6} \pi_2\]
\[\pi_0 = \frac{1}{4} \pi_1 - \frac{5}{4} \pi_2\]
\[\pi_0 + \pi_1 + \pi_2 = 1\]

\[\Rightarrow \pi_1 = \frac{5}{6} \pi_1 + \frac{1}{6} \pi_2\]
\[\pi_1 = \frac{5}{4} \pi_1 - \frac{5}{4} \pi_2\]
\[\pi_0 + \pi_1 + \pi_2 = 1\]

\[\Rightarrow \pi_2 = \frac{5}{6} \pi_1 + \frac{1}{6} \pi_2\]
\[\pi_2 = \frac{1}{4} \pi_1 - \frac{5}{4} \pi_2\]
\[\pi_0 + \pi_1 + \pi_2 = 1\]

\[\Rightarrow \pi_0 = \frac{4}{11}\]
\[\Rightarrow \pi_1 = \frac{17}{44}\]
\[\pi_2 = \frac{14}{44}\]

\[\pi_0 = \frac{1}{\lambda_0} = \frac{4}{11} \Rightarrow \lambda_0 = \frac{11}{4}\]

is the average number of days between two consecutive sunny days.
Problem 4

a) \[ P[D_k = m \mid X_k = i] = \begin{cases} \binom{i}{m} p^m (1-p)^{i-m} & m = 0, 1, \ldots, i \\ 0 & m \geq i+1 \end{cases} \]

b) \[ P[X_{k+1} = j \mid X_k = i] = 0 \quad \text{for} \quad j \geq 1 \]

For \( j \leq i \),

\[ P[X_{k+1} = j \mid X_k = i] = P[X_k - D_k + A_{k+1} = j \mid X_k = i] \]

\[ = \frac{P[X_k - D_k + A_{k+1} = j, X_k = i]}{P[X_k = i]} = \frac{P[A_{k+1} - D_k = j - i, X_k = i]}{P[X_k = i]} \]

\[ = \sum_{m=1}^{\infty} \frac{P[A_{k+1} - D_k = j - i, D_k = m, X_k = i]}{P[X_k = i]} \]

\[ = \sum_{m=1}^{\infty} \frac{P[A_{k+1} = j - i + m]}{P[X_k = i]} \]

\[ = \sum_{m=1}^{\infty} e^{-\lambda} \frac{\lambda^{j-i+m}}{(j-i+m)!} \frac{i!}{m! (i-m)!} p^m (1-p)^{i-m} \]

\[ = e^{-\lambda} \sum_{m=i}^{\infty} \frac{i!}{(j-i+m)! (i-m)!} \lambda^{j-i+m} p^m (1-p)^{i-m} \]

\[ c) \quad X_k = X_0 - D_k + A_k \]

\[ X_0 = 0 \Rightarrow D_k = 0 \]

\[ \Rightarrow X_k = A_k \]

which implies that \( X_2 \) has a Poisson distribution with parameter \( \lambda \).
\[ P[X_i - D_i = k] = P[D_i = X_i - k] = \sum_{k=0}^\infty P[D_i = X_i - k | X_i = l] P[X_i = l] \]
\[ = \sum_{k=0}^\infty P[D_i = l-k | X_i = l] P[X_i = l] \]
\[ = \sum_{k=0}^\infty \left( \frac{\lambda^k}{k!} e^{-\lambda} \right) \frac{\lambda^k}{k!} (1-p)^k \frac{2^l}{l!} e^{-2} \]
\[ = \sum_{k=0}^\infty \left( \frac{\lambda^k}{k!} \right) \frac{\lambda^k}{k!} (1-p)^k \frac{2^l}{l!} e^{-3} \]
\[ = e^{-3} \left( \frac{\lambda}{(1-p)} \right)^k \sum_{k=0}^\infty \frac{(\lambda)^k}{k!} \frac{(1-p)^k}{k!} \frac{2^l}{l!} \]
\[ = e^{-3} \left( \frac{\lambda}{(1-p)} \right)^k \sum_{k=0}^\infty \frac{(\lambda)^k}{k!} \frac{(1-p)^k}{k!} \frac{2^l}{l!} \]
\[ = e^{-3} \left( \frac{\lambda}{(1-p)} \right)^k \]
\[ = e^{-3} \left( \frac{\lambda}{(1-p)} \right)^k e^{-2(1-p)} \]
\[ i.e. \ (X_i - D_i) \text{ has a Poisson distribution with parameter } \lambda(1-p) \]
\[ X_i = (X_i - D_i) + D_i \text{ i.e. } X_i \text{ is the sum of two r.v.s of Poisson distribution with rates } \lambda(1-p) \text{ and } \lambda, \text{ respectively. Consequently, } X_i \text{ has a Poisson distribution of rate } \lambda(1-p) + \lambda = \lambda(1-p). \]

(d) If \( X_0 \sim \text{Poisson}(\nu) \), then from part (b) we can get \((X_0 - D_0) \sim \text{Poisson}(\nu(1-p))\)
It follows that : \( X_1 = (X_0 - D_0) + A_1 \sim \text{Poisson}(\lambda + \nu(1-p)) \).
We would like to have : \( \lambda + \nu(1-p) = \nu \Rightarrow \nu = \frac{\lambda}{1-p} \).
Clearly, the process repeats for \( X_2 = (X_1 - D_1) + A_2 \), etc.
Problem 5

\[ p_0 = \frac{1}{1 + \sum_{m=1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 + \frac{\theta_i}{\mu_i+1} + \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]

\[ = \frac{1}{1 - \frac{(\theta/\mu)^{k+1}}{m=k+1} \frac{\theta_i}{\mu_i+1} - \sum_{m=k+1}^{\infty} \frac{(\theta/\mu)^m}{m!} \frac{\theta_i}{\mu_i+1}} \]
Therefore,

\[
P_m = \begin{cases} 
\left( \frac{\theta}{\mu} \right)^m & \text{for } m \leq k+1 \\
\frac{1 - \frac{2}{\mu} \sum_{j=0}^{m} \frac{2^j}{j!} \left( \frac{1}{k+1} \right)^j}{1 - \frac{2}{\mu} + \frac{2}{\mu} k} \left( e^{\frac{2}{\mu} k} - \sum_{j=0}^{m} \frac{2^j}{j!} \left( \frac{1}{k+1} \right)^j \right) & \text{for } m \geq k+1 
\end{cases}
\]

b) The average queue length \( E[X] \) is

\[
E[X] = \sum_{m=0}^{\infty} m P_m = \left( \sum_{m=0}^{k} m \left( \frac{\theta}{\mu} \right)^m + \sum_{m=k+1}^{\infty} m \left( \frac{\theta}{\mu} \right)^m \frac{k^m}{(m-k)!} \right) \frac{1}{1 - \frac{2}{\mu} + \frac{2}{\mu} k} \left( e^{\frac{2}{\mu} k} - \sum_{m=0}^{k} \frac{2^j}{j!} \left( \frac{1}{k+1} \right)^j \right)
\]

The system time at steady state is

\[ E[S] = \frac{E[X]}{\rho} \]

where \( \rho = 1 - P_0 \) is the throughput of the system.

c) The system remains stable as long as \( \mu > \) average arrival rate, i.e.

\[
\mu > \sum_{n=0}^{k} n \lambda P_n + \sum_{n=k+1}^{\infty} \frac{n}{n} P_n = \lambda \left[ \sum_{n=0}^{k} P_n + \sum_{n=k+1}^{\infty} \frac{P_n}{n} \right]
\]
The computer system can be modeled as an M/M/1/k queue, which is analyzed in section (6.6.1). The blocking probability for any transaction is given by (6.61):

\[ P_b = (1-p) \frac{\rho^k}{k!} \]

where \( \rho = \frac{\mu_1 \mu_2}{\lambda_1} \)

The service rate is \( \mu = \frac{1}{5} \text{ sec}^{-1} \) the rate of the first source transaction is \( \lambda_1 = 4 \text{ min}^{-1} = \frac{1}{15} \text{ sec}^{-1} \) and the rate of the second source transaction is \( \lambda_2 = 3 \text{ min}^{-1} = \frac{1}{20} \text{ sec}^{-1} \).

Thus, \( \rho = \frac{4 \times 3}{15 + 20} = 0.7 \)

and \( \frac{1}{15} + \frac{1}{20} = 0.1 \)

\[ P_b = (1-0.7) \left( \frac{0.7)^k}{1-0.7^{k+1}} \right) = 0.3 \left( \frac{0.7)^k}{1-0.7^{k+1}} \right) \]

The blocking probability for a transaction generated by the first source is \( P_b = \frac{\mu_1}{\lambda_1 \mu_2} \). Hence

\[ P_b \frac{\partial}{\partial \mu_2} \leq 0.1 \Rightarrow 0.3 \left( \frac{0.7)^k}{1-0.7^{k+1}} \right) \frac{1}{15 + \frac{1}{20}} \leq 0.1 \Rightarrow \]

\[ \Rightarrow \frac{3}{10} \frac{0.7^k}{1-0.7^{k+1}} \leq 0.1 \Rightarrow \frac{0.7^k}{1-0.7^{k+1}} \leq 0.1 \Rightarrow \]

\[ \Rightarrow k \log(0.7) \leq \log(0.1) \Rightarrow k \geq \frac{\log(0.7)}{\log(0.1)} \Rightarrow k \geq 2.47 \]

Hence the minimum value of \( k \) is 3.
The blocking probability for a transaction generated by the second source is \( P_b \leq 0.02 \). Hence

\[
P_b \leq 0.02 \Rightarrow 0.3 \times \frac{(0.7)^k}{1 - (0.7)^k} \leq \frac{5}{10} \Rightarrow 0.3 \leq \frac{5}{(10-k)^k}
\]

\[
\Rightarrow \frac{3}{10} \leq \frac{3}{4} \times \frac{(0.7)^k}{1 - (0.7)^k} \leq 0.02 \Rightarrow 9 \times (0.7)^k \leq 1.4 \Rightarrow (0.7)^k \leq \frac{1.4}{9}
\]

\[
(0.7)^k \leq \frac{1.4}{9.98} \Rightarrow k \leq \ln \frac{1.4}{9.98} \Rightarrow k \geq \frac{\ln \frac{1.4}{9.98}}{\ln 0.7} \approx 5.507
\]

Hence the minimum value of \( k \) is 6.

The minimum value of \( k \) for the blocking probability of both sources to satisfy the constraints is 6.

Now the model is an \( M/M/C/1/6 \) queue with arrival rate

\[
\lambda = \frac{1}{15} + \frac{1}{10} = \frac{23}{60}
\]

The average system time \( E[S] \) is given by (6.63):

\[
E[S] = \frac{E[X]}{\lambda (1 - \rho)}
\]

with

\[
E[X] = \frac{1 - \rho}{1 - \rho^k} \left( \frac{k - \rho^k}{1 - \rho^k} - k \rho^k \right) \quad \text{(by (6.62))}
\]

and

\[
\pi_k = (1 - \rho) \frac{\rho^k}{1 - \rho^k} \quad \text{(by (6.61))}
\]

In our case:

\[
k = 6, \lambda = \frac{23}{60}, \mu = \frac{1}{6}, \rho = \frac{23}{6} = \frac{23}{60} = 0.7
\]

\[
E[X] = \frac{0.7^6}{1 - 0.7^6} \left[ \frac{6 - (0.7)^6}{1 - 0.7} - 6 \times (0.7)^6 \right] = \approx 1.87
\]

\[
\pi_k = 0.3 \times \frac{(0.7)^k}{1 - (0.7)^k} = 0.038 \quad \text{and} \quad E[S] = \frac{1.87}{60 (1 - 0.038)} \approx 1.66 \text{ sec.}
\]
Our model is a Jackson network of five nodes. Let's calculate the actual arrival rates $\lambda_i$ at each node $i$:

\[
\begin{align*}
\lambda_1 &= \lambda_3 + \lambda_4 + \lambda_5 = 1 + \lambda_4 \\
\lambda_2 &= \lambda_3 \pi_3 + \lambda_5 \pi_5 = 0.8 \lambda_5 + \lambda_5 \\
\lambda_5 &= \lambda_3 \pi_5 = \lambda_2 \\
\lambda_4 &= \lambda_3 \pi_4 + \lambda_5 \pi_5 = 0.8 \lambda_1 + 1.0 \lambda_5 \\
\lambda_5 &= \lambda_3 \pi_5 = 0.8 \lambda_5
\end{align*}
\]

\[
\begin{align*}
\lambda_1 &= \frac{5}{3} \\
\lambda_2 &= \frac{5}{3} \\
\lambda_3 &= \frac{5}{3} \\
\lambda_4 &= \frac{5}{3} \\
\lambda_5 &= \frac{1}{3}
\end{align*}
\]

a) Each node $M_i$ can be treated as an $M/M/1$ queuing system with arrival rate $\lambda_i$ and service rate $\mu_i$.

Hence,

\[ p_1 = \frac{\lambda_1}{\mu_1} = \frac{5/3}{2} = \frac{5}{6} \quad \text{and} \quad p_2 = \frac{\lambda_2}{\mu_2} = \frac{5/3}{\mu_4} = \frac{20}{21} \]

\[ p_3 = \frac{\lambda_3}{\mu_3} = \frac{5/3}{2} = \frac{5}{3} \quad \text{and} \quad p_4 = \frac{\lambda_4}{\mu_4} = \frac{2/3}{1} = \frac{2}{3} \quad \text{and} \quad p_5 = \frac{\lambda_5}{\mu_5} = \frac{1/3}{0.5} = \frac{2}{3} \]

b) The throughput of the system is

\[ \rho = \lambda_3 \pi_3 = \frac{5}{3} \times 0.6 = 1 \]

c) If $X_i$ is the queue length at node $M_i$, $i=1,2,...,5$, then the average total number of customers in the system, $E[X]$, is

\[ E[X] = \sum_{i=1}^{5} E[X_i] = \sum_{i=1}^{5} \frac{\lambda_i}{1-\rho_i} = \frac{5}{1-5/6} + \frac{20/21}{1-20/21} + \frac{25/33}{1-25/33} + \frac{2/3}{1-2/3} + \frac{2/3}{1-2/3} \]

\[ = 5 + 20 + \frac{25}{8} + 2 + 2 = 32.125 \]

The arrival rate $\lambda$ into the system is

\[ \lambda = \lambda_3 \pi_3 = 1 \]

By Little's law, the average system time $E[S]$ is

\[ E[S] = \frac{E[X]}{\lambda} = \frac{32.125}{1} = 32.125 \]
$$P(X_2 > 3) = 1 - P(X_2 = 0) - P(X_2 = 1) - P(X_2 = 2) - P(X_2 = 3)$$

$$= 1 - (1-p_2) - (1-p_2) p_2 - (1-p_2) p_2^2 - (1-p_2) p_2^3$$

$$= 1 - \frac{1}{2.1} - \frac{1}{2.1} \frac{20}{21} - \frac{1}{2.1} \left( \frac{20}{21} \right)^2 - \frac{1}{2.1} \left( \frac{20}{21} \right)^3$$

$$\approx 0.823$$
Problem 8

\[ P_1 = p \quad P_2 = (1-p) \]
\[ P_1 = \frac{\beta_1}{\mu_1} \quad P_2 = \frac{\beta_2}{\mu_2} \]
\[ P_2 = \frac{\beta_2}{\mu_2} \quad P_2 = \frac{u_2}{\mu_2} = \frac{u_2-eta_2}{\mu_2} \]

The average system time for queue 1 is:
\[ E[S_1] = \frac{1/\mu_1}{1-P_1} = \frac{1/\mu_1}{1-\frac{\beta_1}{\mu_1}} = \frac{1}{1-\frac{\beta_1}{\mu_1}} \]

and for queue 2:
\[ E[S_2] = \frac{1/\mu_2}{1-P_2} = \frac{1/\mu_2}{1-\frac{\beta_2}{\mu_2}} = \frac{1}{1-\frac{\beta_2}{\mu_2}} \]

The average system time for the whole system is:
\[ E[S] = pE[S_1] + (1-p)E[S_2] \]
\[ = \frac{p}{1-\frac{\beta_1}{\mu_1}} + \frac{1-p}{1-\frac{\beta_2}{\mu_2}} \]

The value of \( p \) that minimizes \( E[S] \) is determined as follows:
\[ \frac{dE[S]}{dp} = 0 \Rightarrow \frac{\mu_1-\beta_1-\beta_2(1-p)}{(1-\frac{\beta_1}{\mu_1})^2} + \frac{-\mu_2(1-p) + \frac{\beta_2}{\mu_2}}{[\mu_2-(1-\frac{\beta_1}{\mu_1})]^2} = 0 \]

\[ \Rightarrow \mu_1^2 - \mu_1(1-p)\beta_1 - \mu_1^2 - \mu_1^2(p_1 - \beta_2) = 0 \]

\[ \Rightarrow \mu_1^2 \mu_2(1-p)^2 - 2\gamma \mu_2(1-p) = 0 \]

\[ \Rightarrow \mu_1^2 \mu_2^2(1-p)^2 - 2\gamma \mu_2(1-p) = 0 \]

\[ \Rightarrow \mu_1^2 \mu_2^2(1-p)^2 - 2\gamma \mu_2(1-p) = 0 \]

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\[ \Rightarrow \mu_1^2 \mu_2^2(1-p)^2 - 2\gamma \mu_2(1-p) = 0 \]

\[ \Rightarrow \mu_1^2 \mu_2^2(1-p)^2 - 2\gamma \mu_2(1-p) = 0 \]

For \( \gamma = 2, \mu_1 = 1, \mu_2 = 1.5 \), we get:
\[ 4(1-1.5)p^2 + 4(1.5+1.5-2) = 10p^2 + 2.25 + 4 - 1.5 = 0 \]
\[ \Rightarrow -2p^2 + 4p - 1.25 = 0 \Rightarrow 2p^2 - 4p + 1.25 = 0 \Rightarrow \]
\[ p_1 = \frac{1 + \sqrt{1}}{4} = 0.388 \]
\[ p_2 = 1 - \frac{\sqrt{1}}{4} = 0.388 \]
b) In this case, we have:

\[ E[S_1] = \frac{1}{\mu_2 - \mu_2^2} = \frac{1}{1-2p} \]  as before, but to evaluate \( E[S_2] \) we must

set \( g=0 \) in the Pollaczek-Khinchin formula (6.120):

\[
E[S_2] = \frac{1}{\mu_2} - \frac{\mu_2}{2(1-p)} = \frac{1}{\mu_2} - \frac{4(1-p)}{\mu_2} - \frac{2(1-p)}{3(1-p)}
\]

\[
= \frac{1}{\mu_2 - (1-p)^2} - \frac{2\mu_2[1-p(1-p)]}{4(1-p)}
\]

\[
= \frac{1}{\mu_2 - (1-p)^2} - \frac{2(1-p)}{3(1-p)}
\]

\[
= \frac{2}{4p-1} - \frac{2(1-p)}{3(4p-1)}
\]

\[
= \frac{2(1+2p)}{3(4p-1)}
\]

\[
E[S] = \frac{p}{1-2p} + \frac{2(1-p)(1+2p)}{3(4p-1)}
\]

To calculate \( p \) that minimizes \( E[S] \):

\[
\frac{dE[S]}{dp} = \frac{1}{(1-2p)^2} + \frac{-4p^2 + 24p - 30}{9(4p-1)^2} = 0
\]

\[ q(4p-1)(1-2p)^2(-4p^2 + 24p - 30) = 0 \]

\[ 144p^2 - 72p + 9 - 48p^2 + 24p - 30 = 0 \]

\[ -192p^4 + 288p^3 - 120p^2 + 192p^3 - 96p^2 + 120p = 0 \]

\[ -192p^4 + 288p^3 - 120p^2 + 72p - 18 = 0 \]

\[ -64p^4 + 96p^3 - 40p^2 + 24p - 7 = 0 \]

\[ \Rightarrow p = 0.37 \quad \text{(numerically)} \]