Markov Chains

Summary

- Markov Chains
- Discrete Time Markov Chains
  - Homogeneous and non-homogeneous Markov chains
  - Transient and steady state Markov chains
- Continuous Time Markov Chains
  - Homogeneous and non-homogeneous Markov chains
  - Transient and steady state Markov chains
Markov Processes

- Recall the definition of a Markov Process:
  - The future a process does not depend on its past, only on its present.
  \[
  \Pr \left\{ X(t_{k+1}) \leq x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_0) = x_0 \right\} = \Pr \left\{ X(t_{k+1}) \leq x_{k+1} \mid X(t_k) = x_k \right\}
  \]

- Since we are dealing with "chains", $X(t)$ can take discrete values from a finite or a countable infinite set.

- For a discrete-time Markov chain, the notation is also simplified to
  \[
  \Pr \left\{ X_{k+1} = x_{k+1} \mid X_k = x_k, \ldots, X_0 = x_0 \right\} = \Pr \left\{ X_{k+1} = x_{k+1} \mid X_k = x_k \right\}
  \]

- Where $X_k$ is the value of the state at the $k$th step.

Chapman-Kolmogorov Equations

- Define the one-step transition probabilities
  \[
  p_{ij} (k) = \Pr \left\{ X_{k+1} = j \mid X_k = i \right\}
  \]

- Clearly, for all $i$, $k$, and all feasible transitions from state $i$
  \[
  \sum_{j \in \Gamma(i)} p_{ij} (k) = 1
  \]

- Define the $n$-step transition probabilities
  \[
  p_{ij} (k, k+n) = \Pr \left\{ X_{k+n} = j \mid X_k = i \right\}
  \]
Using total probability

\[ p_{ij}(k, k+n) = \sum_{r=1}^{R} \Pr\{X_{k+n} = j \mid X_u = r, X_k = i\} \Pr\{X_u = r \mid X_k = i\} \]

Using the memoryless property of Markov chains

\[ \Pr\{X_{k+n} = j \mid X_u = r, X_k = i\} = \Pr\{X_{k+n} = j \mid X_u = r\} \]

Therefore, we obtain the Chapman-Kolmogorov Equation

\[ p_{ij}(k, k+n) = \sum_{r=1}^{R} p_{ir}(k, u) p_{rj}(u, k+n), \quad k \leq u \leq k+n \]

Matrix Form

Define the matrix

\[ H(k, k+n) = \left[ p_{ij}(k, k+n) \right] \]

We can re-write the Chapman-Kolmogorov Equation

\[ H(k, k+n) = H(k, u) H(u, k+n) \]

Choose, \( u = k+n-1 \), then

\[ H(k, k+n) = H(k, k+n-1) H(k+n-1, k+n) = H(k, k+n-1) P(k+n-1) \]

*Forward* Chapman-Kolmogorov

One step transition probability
Matrix Form

Choose, \( u = k+1 \), then

\[
H(k, k+n) = H(k, k+1)H(k+1, k+n) = P(k)H(k+1, k+n)
\]

**Backward Chapman-Kolmogorov**

One step transition probability

Homogeneous Markov Chains

- The one-step transition probabilities are independent of time \( k \).
  \[
P(k) = P \quad \text{or} \quad [p_{ij}] = \Pr\{X_{k+1} = j \mid X_k = i\}
\]

- Even though the one step transition is independent of \( k \), this does not mean that the joint probability of \( X_{k+1} \) and \( X_k \) is also independent of \( k \)
  - Note that

\[
\Pr\{X_{k+1} = j, X_k = i\} = \Pr\{X_{k+1} = j \mid X_k = i\}\Pr\{X_k = i\} = p_{ij}\Pr\{X_k = i\}
\]
Example

- Consider a two transmitter (Tx) communication system where, time is divided into time slots and that operates as follows
  - At most one packet can arrive during any time slot and this can happen with probability $\alpha$.
  - Packets are transmitted by whichever transmitter is available, and if both are available then the packet is given to Tx 1.
  - If both transmitters are busy, then the packet is lost
  - When a Tx is busy, it can complete the transmission with probability $\beta$ during any one time slot.
  - If a packet is submitted during a slot when both transmitters are busy but at least one Tx completes a packet transmission, then the packet is accepted (departures occur before arrivals).

- Describe the Markov Chain that describe this model.

Example: Markov Chain

- For the State Transition Diagram of the Markov Chain, each transition is simply marked with the transition probability

```
p_{00} = (1 - \alpha) 
\begin{align*}
p_{10} &= \beta (1 - \alpha) 
p_{20} &= \beta^2 (1 - \alpha) 
\end{align*}
\begin{align*}
p_{01} &= \alpha 
p_{11} &= (1 - \beta)(1 - \alpha) + \alpha \beta 
p_{21} &= \beta^2 \alpha + 2 \beta (1 - \beta)(1 - \alpha) 
\end{align*}
\begin{align*}
p_{02} &= 0 
p_{12} &= \alpha (1 - \beta) 
p_{22} &= (1 - \beta)^2 + 2 \alpha \beta (1 - \beta)
\end{align*}
```
Example: Markov Chain

Suppose that $\alpha = 0.5$ and $\beta = 0.7$, then,

$$
\begin{bmatrix}
0.5 & 0.5 & 0 \\
0.35 & 0.5 & 0.15 \\
0.245 & 0.455 & 0.3
\end{bmatrix}
$$

State Holding Times

Suppose that at point $k$, the Markov Chain has transitioned into state $X_k = i$. An interesting question is how long it will stay at state $i$.

Let $V(i)$ be the random variable that represents the number of time slots that $X_k = i$.

We are interested in the quantity $\Pr\{V(i) = n\}$

$$
\Pr\{V(i) = n\} = \Pr\{X_{k+n} \neq i, X_{k+n-1} = i, \ldots, X_{k+1} = i \mid X_k = i\} \\
= \Pr\{X_{k+n} \neq i \mid X_{k+n-1} = i, \ldots, X_k = i\} \times \\
\Pr\{X_{k+n-1} = i, \ldots, X_{k+1} = i \mid X_k = i\} \times \\
\Pr\{X_{k+n-1} = i \mid X_{k+n-2}, \ldots, X_k = i\} \times \\
\Pr\{X_{k+n-2} = i, \ldots, X_{k+1} = i \mid X_k = i\}
$$
State Holding Times

\[
\Pr \{ V(i) = n \} = \Pr \{ X_{k+n} \neq i \mid X_{k+n-1} = i \} \times \\
\Pr \{ X_{k+n-1} = i \mid X_{k+n-2} \ldots, X_k = i \} \times \\
\Pr \{ X_{k+n-2} = i, \ldots, X_{k+1} = i \mid X_k = i \} \\
= (1 - p_{ii}) \Pr \{ X_{k+n-1} = i \mid X_{k+n-2} = i \} \times \\
\Pr \{ X_{k+n-2} = i \mid X_{k+n-3} = i, \ldots, X_k = i \} \\
\Pr \{ X_{k+n-3} = i, \ldots, X_{k+1} = i \mid X_k = i \} \\
\Pr \{ V(i) = n \} = (1 - p_{ii}) p_{ii}^{n-1}
\]

- This is the Geometric Distribution with parameter \( p_{ii} \).
- \( V(i) \) has the memoryless property.

State Probabilities

- An interesting quantity we are usually interested in is the probability of finding the chain at various states, i.e., we define

\[
\pi_i(k) \equiv \Pr \{ X_k = i \}
\]

- For all possible states, we define the vector

\[
\pi(k) = [\pi_0(k), \pi_1(k) \ldots]
\]

- Using total probability we can write

\[
\pi_i(k) = \sum_j \Pr \{ X_k = i \mid X_{k-1} = j \} \Pr \{ X_{k-1} = j \}
\]

\[
= \sum_j p_{ij}(k) \pi_j(k-1)
\]

- In vector form, one can write

\[
\pi(k) = \pi(k-1) P(k)
\]

Or, if homogeneous Markov Chain

\[
\pi(k) = \pi(k-1) P
\]
State Probabilities Example

- Suppose that
  \[
  \mathbf{P} = \begin{bmatrix}
  0.5 & 0.5 & 0 \\
  0.35 & 0.5 & 0.15 \\
  0.245 & 0.455 & 0.3
  \end{bmatrix}
  \]  
  with  \( \pi(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \)

- Find \( \pi(k) \) for \( k = 1, 2, \ldots \)

\[
\pi(1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\
0.35 & 0.5 & 0.15 \\
0.245 & 0.455 & 0.3
\end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}
\]

- *Transient* behavior of the system: MCTransient.m

- In general, the transient behavior is obtained by solving the difference equation

\[
\pi(k) = \pi(k-1)\mathbf{P}
\]

Classification of States

- Definitions
  - State \( j \) is **reachable** from state \( i \) if the probability to go from \( i \) to \( j \) in \( n > 0 \) steps is greater than zero (State \( j \) is reachable from state \( i \) if in the state transition diagram there is a path from \( i \) to \( j \)).
  - A subset \( S \) of the state space \( X \) is **closed** if \( p_{ij}=0 \) for every \( i \in S \) and \( j \notin S \)
  - A state \( i \) is said to be **absorbing** if it is a single element closed set.
  - A closed set \( S \) of states is **irreducible** if any state \( j \in S \) is reachable from every state \( i \in S \).
  - A Markov chain is said to be **irreducible** if the state space \( X \) is irreducible.
Example

- Irreducible Markov Chain

  ![Irreducible Markov Chain Diagram]

- Reducible Markov Chain

  ![Reducible Markov Chain Diagram]

  Absorbing State
  
  Closed irreducible set

Transient and Recurrent States

- Hitting Time \( T_{ij} = \min \{ k > 0 : X_0 = i, X_k = j \} \)

- Recurrence Time \( T_{ii} \) is the first time that the MC returns to state \( i \).

  Let \( \rho_i \) be the probability that the state will return back to \( i \) given it starts from \( i \). Then,

  \[
  \rho_i = \sum_{k=1}^{\infty} \Pr\{T_{ii} = k\}
  \]

  The event that the MC will return to state \( i \) given it started from \( i \) is equivalent to \( T_{ii} < \infty \), therefore we can write

  \[
  \rho_i = \sum_{k=1}^{\infty} \Pr\{T_{ii} = k\} = \Pr\{T_{ii} < \infty\}
  \]

- A state is recurrent if \( \rho_i = 1 \) and transient if \( \rho_i < 1 \)
Theorems

- If a Markov Chain has finite state space, then at least one of the states is recurrent.

- If state $i$ is recurrent and state $j$ is reachable from state $i$ then, state $j$ is also recurrent.

- If $S$ is a finite closed irreducible set of states, then every state in $S$ is recurrent.

Positive and Null Recurrent States

- Let $M_i$ be the mean recurrence time of state $i$

\[ M_i = E[T_{ii}] = \sum_{k=1}^{\infty} k \Pr\{T_{ii} = k\} \]

- A state is said to be positive recurrent if $M_i < \infty$. If $M_i = \infty$ then the state is said to be null-recurrent.

- Theorems
  - If state $i$ is positive recurrent and state $j$ is reachable from state $i$ then, state $j$ is also positive recurrent.
  - If $S$ is a closed irreducible set of states, then every state in $S$ is positive recurrent or, every state in $S$ is null recurrent, or, every state in $S$ is transient.
  - If $S$ is a finite closed irreducible set of states, then every state in $S$ is positive recurrent.
Suppose that the structure of the Markov Chain is such that state $i$ is visited after a number of steps that is an integer multiple of an integer $d > 1$. Then the state is called \textbf{periodic} with period $d$.

If no such integer exists (i.e., $d = 1$) then the state is called \textbf{aperiodic}.

\textbf{Example}

\begin{align*}
\begin{bmatrix}
0 & 1 & 0 \\
0.5 & 0 & 0.5 \\
0 & 1 & 0
\end{bmatrix}
\end{align*}
Steady State Analysis

- Recall that the probability of finding the MC at state $i$ after the $k$th step is given by
  $$\pi_i(k) \equiv \Pr\{X_k = i\} \quad \pi(k) = [\pi_0(k), \pi_1(k), \ldots]$$
- An interesting question is what happens in the “long run”, i.e.,
  $$\pi_i \equiv \lim_{k \to \infty} \pi_i(k)$$
- This is referred to as **steady state** or **equilibrium** or **stationary state** probability

Questions:
- Do these limits exists?
- If they exist, do they converge to a legitimate probability distribution, i.e., $\sum \pi_i = 1$
- How do we evaluate $\pi_j$, for all $j$.

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Steady State Analysis

- Recall the recursive probability
  $$\pi(k+1) = \pi(k)P$$
- If steady state exists, then $\pi(k+1) = \pi(k)$, and therefore the steady state probabilities are given by the solution to the equations
  $$\pi = \pi P \quad \text{and} \quad \sum_i \pi_i = 1$$
- For Irreducible Markov Chains the presence of periodic states prevents the existence of a steady state probability

**Example:** periodic.m

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} \quad \pi(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
Steady State Analysis

- **THEOREM**: If an irreducible aperiodic Markov chain consists of *positive recurrent* states, a unique stationary state probability vector \( \pi \) exists such that \( \pi_j > 0 \) and

\[
\pi_j = \lim_{k \to \infty} \pi_j(k) = \frac{1}{M_j}
\]

where \( M_j \) is the mean recurrence time of state \( j \)

- The steady state vector \( \pi \) is determined by solving

\[
\pi = \pi P \quad \text{and} \quad \sum \pi_i = 1
\]

- Ergodic Markov chain.

Birth-Death Example

Thus, to find the steady state vector \( \pi \) we need to solve

\[
\pi = \pi P \quad \text{and} \quad \sum \pi_i = 1
\]
Birth-Death Example

In other words

\[ \pi_0 = \pi_0 p + \pi_1 p \]
\[ \pi_j = \pi_{j-1} (1-p) + \pi_{j+1} p, \quad j = 1, 2, \ldots \]

Solving these equations we get

\[ \pi_1 = \frac{1-p}{p} \pi_0 \]
\[ \pi_2 = \left( \frac{1-p}{p} \right)^2 \pi_0 \]

In general

\[ \pi_j = \left( \frac{1-p}{p} \right)^j \pi_0 \]

Summing all terms we get

\[ \pi_0 \sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = 1 \Rightarrow \pi_0 = \frac{1}{\sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i} \]

Therefore, for all states \( j \) we get

\[ \pi_j = \left( \frac{1-p}{p} \right)^j \frac{1}{\sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i} \]

If \( p < 1/2 \), then

\[ \sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = \infty \]

\[ \Rightarrow \pi_j = 0, \quad \text{for all } j \]

All states are transient

If \( p > 1/2 \), then

\[ \sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = \frac{p}{2p-1} > 0 \]

\[ \Rightarrow \pi_j = \frac{2p-1}{p} \left( \frac{1-p}{p} \right)^j, \quad \text{for all } j \]

All states are positive recurrent
Birth-Death Example

- If \( p = 1/2 \), then
  \[
  \sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = \infty \quad \Rightarrow \pi_j = 0, \quad \text{for all } j
  \]
  All states are \textit{null recurrent}

Continuous-Time Markov Chains

- In this case, transitions can occur at \textit{any} time
- Recall the Markov (memoryless) property
  \[
  \Pr \{ X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_0) = x_0 \} = \Pr \{ X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k \}
  \]
  where \( t_1 < t_2 < \ldots < t_k \)
- Recall that the Markov property implies that
  - \( X(t_{k+1}) \) depends only on \( X(t_k) \) (state memory)
  - It does not matter how long the state is at \( X(t_k) \) (age memory).
- The transition probabilities now need to be defined for every time instant as \( p_{ij}(t) \), i.e., the probability that the MC transitions from state \( i \) to \( j \) at time \( t \).
Transition Function

- Define the transition function
  \[ p_{ij}(s,t) \equiv \Pr\{X(t) = j \mid X(s) = i\}, \quad s \leq t \]

- The continuous-time analogue of the Chapman-Kolmogorov equation is
  \[ p_{ij}(s,t) \equiv \sum_r \Pr\{X(t) = j \mid X(u) = r, X(s) = i\} \Pr\{X(u) = r \mid X(s) = i\} \]

- Using the memoryless property
  \[ p_{ij}(s,t) \equiv \sum_r \Pr\{X(t) = j \mid X(u) = r\} \Pr\{X(u) = r \mid X(s) = i\} \]

- Define \( H(s,t) = [p_{ij}(s,t)], i,j = 1,2,\ldots \) then
  \[ H(s,t) = H(s,u)H(u,t), \quad s \leq u \leq t \]

  \( \square \) Note that \( H(s,s) = I \)

Transition Rate Matrix

- Consider the Chapman-Kolmogorov for \( s \leq t \leq t + \Delta t \)
  \[ H(s,t + \Delta t) = H(s,t)H(t,t + \Delta t) \]

- Subtracting \( H(s,t) \) from both sides and dividing by \( \Delta t \)
  \[ \frac{H(s,t + \Delta t) - H(s,t)}{\Delta t} = \frac{H(s,t)(H(t,t + \Delta t) - I)}{\Delta t} \]

- Taking the limit as \( \Delta t \to 0 \)
  \[ \frac{\partial H(s,t)}{\partial t} = H(s,t)Q(t) \]
  
  where the transition rate matrix \( Q(t) \) is given by
  \[ Q(t) = \lim_{\Delta t \to 0} \frac{H(t,t + \Delta t) - I}{\Delta t} \]
Homogeneous Case

- In the homogeneous case, the transition functions do not depend on \( s \) and \( t \), but only on the difference \( t-s \) thus 
  \[ p_{ij}(s,t) = p_{ij}(t-s) \]

- It follows that 
  \[ H(s,t) = H(t-s) \equiv P(\tau) \]
  and the transition rate matrix

  \[ Q(t) = \lim_{\Delta t \to 0} \frac{H(t,t+\Delta t)-I}{\Delta t} = \lim_{\Delta t \to 0} \frac{H(\Delta t)-I}{\Delta t} = Q, \text{ constant} \]

- Thus
  \[ \frac{\partial P(t)}{\partial t} = P(t)Q \text{ with } p_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \Rightarrow P(t) = e^{Qt} \]

State Holding Time

- The time the MC will spend at each state is a random variable with distribution

  \[ G_i(t) = 1 - e^{-\Lambda_i} \]

  where

  \[ \Lambda_i = \sum_{j \in \Gamma(i)} \lambda_j \]

- Explain why…
Recall that
\[ \frac{\partial P(t)}{\partial t} = P(t)Q \Rightarrow \frac{\partial p_{ij}(t)}{\partial t} = \sum_r p_{ir}(t)q_{rj} \]

First consider the \( q_{ij}, \ i \neq j \), thus the above equation can be written as
\[ \frac{\partial p_{ij}(t)}{\partial t} = p_{ii}(t)q_{ij} + \sum_{r \neq i} p_{ir}(t)q_{rj} \]

Evaluating this at \( t = 0 \), we get that
\[ \frac{\partial p_{ij}(t)}{\partial t} \bigg|_{t=0} = q_{ij} \quad \Rightarrow \quad p_{ij}(0) = 0 \text{ for all } i \neq j \]

The event that will take the state from \( i \) to \( j \) has exponential residual lifetime with rate \( \lambda_{ij} \), therefore, given that in the interval \((t, t + \tau)\) one event has occurred, the probability that this transition will occur is given by
\[ G_{ij}(\tau) = 1 - \exp\{-\lambda_{ij}\tau\} \]

Since \( G_{ij}(\tau) = 1 - \exp\{-\lambda_{ij}\tau\} \).
\[ \frac{\partial p_{ij}(\tau)}{\partial \tau} \bigg|_{\tau=0} = q_{ij} = \lambda_{ij} e^{\lambda_{ij}\tau} \bigg|_{\tau=0} = \lambda_{ij} \]

In other words \( q_{ij} \) is the rate of the Poisson process that activates the event that makes the transition from \( i \) to \( j \).

Next, consider the \( q_{jj} \), thus
\[ \frac{\partial p_{jj}(t)}{\partial t} = p_{jj}(t)q_{jj} + \sum_{r \neq j} p_{ir}(t)q_{rj} \]

Evaluating this at \( t = 0 \), we get that
\[ \frac{\partial p_{jj}(t)}{\partial t} \bigg|_{t=0} = q_{jj} \iff \frac{\partial}{\partial t} \left[ 1 - p_{jj}(t) \right] \bigg|_{t=0} = -q_{jj} \]

Probability that chain leaves state \( j \).
Transition Rate Matrix $\mathbf{Q}$.

- The event that the MC will transition out of state $i$ has exponential residual lifetime with rate $\Lambda(i)$, therefore, the probability that an event will occur in the interval $(t, t+\tau)$ given by $G_i(\tau) = 1 - \exp\{-\Lambda(i)\tau\}$.

$$-q_{jj} = \Lambda(i) e^{-\Lambda(i)\tau}\bigg|_{\tau=0} = \Lambda(i)$$

- Note that for each row $i$, the sum

$$\sum_j q_{ij} = 0$$

Transition Probabilities $\mathbf{P}$.

- Suppose that state transitions occur at random points in time $T_1 < T_2 < \ldots < T_k < \ldots$

- Let $X_k$ be the state after the transition at $T_k$

- Define

$$P_{ij} = \Pr\{X_{k+1} = j \mid X_k = i\}$$

- Recall that in the case of the superposition of two or more Poisson processes, the probability that the next event is from process $j$ is given by $\lambda_j/\Lambda$.

- In this case, we have

$$P_{ij} = \frac{q_{ij}}{-q_{ii}}, i \neq j \quad \text{and} \quad P_{ii} = 0$$
Example

- Assume a transmitter where packets arrive according to a Poisson process with rate $\lambda$.
- Each packet is processed using a First In First Out (FIFO) policy.
- The transmission time of each packet is exponential with rate $\mu$.
- The transmitter has buffer to store up to two packets that wait to be transmitted.
- Packets that find the buffer full are lost.
- Draw the state transition diagram.
- Find the Rate Transition Matrix $Q$.
- Find the State Transition Matrix $P$

Example

The rate transition matrix is given by

$$Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 \\
\mu & - (\lambda + \mu) & \lambda & 0 \\
0 & \mu & - (\lambda + \mu) & \lambda \\
0 & 0 & \mu & - \mu
\end{bmatrix}$$

The state transition matrix is given by

$$P = \frac{1}{(\lambda + \mu)} \begin{bmatrix}
0 & (\lambda + \mu) & 0 & 0 \\
\mu & 0 & \lambda & 0 \\
0 & \mu & 0 & \lambda \\
0 & 0 & (\lambda + \mu) & 0
\end{bmatrix}$$
State Probabilities and Transient Analysis

- Similar to the discrete-time case, we define
  \[ \pi_j(t) \equiv \Pr \{ X(t) = j \} \]
- In vector form
  \[ \mathbf{\pi}(t) = [\pi_1(t), \pi_2(t), \ldots] \]
- With initial probabilities
  \[ \mathbf{\pi}(0) = [\pi_1(0), \pi_2(0), \ldots] \]
- Using our previous notation (for homogeneous MC)
  \[ \mathbf{\pi}(t) = \mathbf{\pi}(0) \mathbf{P}(t) = \mathbf{\pi}(0) e^{Qt} \]

- Obtaining a general solution is not easy!

- Differentiating with respect to \( t \) gives us more “inside”
  \[ \frac{d\mathbf{\pi}(t)}{dt} = \mathbf{\pi}(t) \mathbf{Q} \iff \frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t) \]

“Probability Fluid” view

- We view \( \pi_j(t) \) as the level of a “probability fluid” that is stored at each node \( j \) (0=empty, 1=full).

\[ \frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t) \]

Change in the probability fluid

\[ -q_{jj} = \sum_{r \neq j} q_{jr} \]

- Inflow
- Outflow

Inflow

Outflow
Steady State Analysis

- Often we are interested in the “long-run” probabilistic behavior of the Markov chain, i.e.,
  \[ \pi_j = \lim_{t \to \infty} \pi_j(t) \]
- These are referred to as **steady state probabilities** or **equilibrium state probabilities** or **stationary state probabilities**
- As with the discrete-time case, we need to address the following questions
  - Under what conditions do the limits exist?
  - If they exist, do they form legitimate probabilities?
  - How can we evaluate these limits?

**Theorem:** In an irreducible continuous-time Markov Chain consisting of positive recurrent states, a unique stationary state probability vector \( \pi \) with
  \[ \pi_j = \lim_{t \to \infty} \pi_j(t) \]
- These vectors are independent of the initial state probability and can be obtained by solving
  \[ \pi Q = 0 \quad \text{and} \quad \sum_j \pi_j = 1 \]
- Using the “probability fluid” view

\[ 0 = q_{jj} \pi_j(t) + \sum_{i \neq j} q_{ij} \pi_i(t) \]
Example

For the previous example, with the above transition function, what are the steady state probabilities

\[ \mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 \\ 0 & \mu & -\lambda + \mu & \lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix} = \mathbf{0} \]

\[ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \]

Example

The solution is obtained

\[ -\lambda \pi_0 + \mu \pi_1 = 0 \quad \Rightarrow \pi_1 = \frac{\lambda}{\mu} \pi_0 \]

\[ \lambda \pi_0 - (\lambda + \mu) \pi_1 + \mu \pi_2 = 0 \quad \Rightarrow \pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0 \]

\[ \lambda \pi_1 - (\lambda + \mu) \pi_2 + \mu \pi_3 = 0 \quad \Rightarrow \pi_3 = \left(\frac{\lambda}{\mu}\right)^3 \pi_0 \]

\[ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \quad \Rightarrow \quad \pi_0 = \frac{1}{1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3} \]
Birth-Death Chain

Find the steady state probabilities

Similarly to the previous example,

\[
Q = \begin{bmatrix}
-\lambda_0 & \lambda_0 & 0 & \cdots \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

And we solve

\[
\pi Q = 0 \quad \text{and} \quad \sum_{i=0}^{\infty} \pi_i = 1
\]

Example

The solution is obtained

\[-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \quad \Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0\]

\[\lambda_0 \pi_0 - (\lambda_1 + \mu_1) \pi_1 + \mu_2 \pi_2 = 0 \quad \Rightarrow \pi_2 = \left(\frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}\right) \pi_0\]

In general

\[\lambda_{j-1} \pi_{j-1} - (\lambda_j + \mu_j) \pi_j + \mu_{j+1} \pi_{j+1} = 0 \quad \Rightarrow \pi_{j+1} = \left(\frac{\lambda_0 \cdots \lambda_j}{\mu_1 \cdots \mu_{j+1}}\right) \pi_0\]

Making the sum equal to 1

\[
\pi_0 \left(1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}\right)\right) = 1
\]

Solution exists if

\[S = 1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}\right) < \infty\]
Uniformization of Markov Chains

- In general, discrete-time models are easier to work with, and computers (that are needed to solve such models) operate in discrete-time.
- Thus, we need a way to turn continuous-time to discrete-time Markov Chains.

Uniformization Procedure

- Recall that the total rate out of state $i$ is $-q_{ii} = A(i)$. Pick a uniform rate $\gamma$ such that $\gamma \geq A(i)$ for all states $i$.
- The difference $\gamma - A(i)$ implies a “fictitious” event that returns the MC back to state $i$ (self loop).

Uniformization Procedure

- Let $P^{U}_{ij}$ be the transition probability from state $i$ to state $j$ for the discrete-time uniformized Markov Chain, then

$$
P^{U}_{ij} = \begin{cases} 
\frac{q_{ij}}{\gamma} & \text{if } i \neq j \\
\frac{\gamma - \sum_{j \neq i} q_{ij}}{\gamma} & \text{if } i = j 
\end{cases}
$$