# Analysis of Optimal Sets of Survivable Paths in Undirected Simple Graph Applicable for Optical Networks

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**Abstract**

In this work we introduce and describe *optimal sets of survivable paths* defined on an undirected simple graph $G$. An optimal set of survivable paths in $G$ corresponds to a set of mesh-restored lightpaths in an optical network that minimizes the number of optical channels. We present several fundamental properties of optimal sets of survivable paths in $G$.

*Keywords:* Shortest paths, Network Survivability, Routing.

## 1 Introduction

Mesh architecture has emerged as an important alternative for an efficient optical network solution [5]. An optical mesh network consists of optical switches interconnected by fiber links, and containing optical circuits called *lightpaths*. In addition, each link contains a number of *optical channels*. A lightpath

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consumes exactly one optical channel on every link on its route. There are different types of lightpaths. In particular, a *mesh-restored lightpath* has a working route and a diversely routed backup route. The wavelength channels on the working route of the mesh-restored lightpath are dedicated for that lightpath and carry user traffic under normal operating conditions. The optical channels on the backup route for the mesh-restored lightpath are shared among different mesh-restored lightpaths. Optical channels are shared in such a way that every mesh-restored lightpath can recover from any single link failure on its working route. In this work we model a mesh optical network with a simple undirected graph $G$ that contains mesh-restored lightpaths only.

A *survivable path* between a pair of vertices in $G$ is a pair of edge-disjoint paths consisting of a working path and a protection path. Protection paths share edges in such a manner as to provide guaranteed recovery upon the failure of any single edge [1]. An ***optimal set of survivable paths*** in $G$ corresponds to a set of mesh-restored lightpaths in an optical network that minimizes the number of optical channels. Hence, it is a practical and important entity because it corresponds to the optimal routing and network design problem in the real world, which is *NP-Complete* [3, 4].

We introduce the following assumptions and definitions to demonstrate the properties of the optimal sets of survivable paths. Throughout the paper we assume the cost of all edges in $G$ to be equal. A 2-edge-connected graph $G$ is *k-optimal* if for any combination of paths defined between end-vertices of distance not less than $k$ in $G$, there exists a corresponding optimal set of survivable paths defined between the same end-vertices with all shortest working paths. We call two survivable paths *distinct* if they are not defined for the same pair of end-vertices. If $G$ is Hamiltonian, then *C(G)* denotes a Hamiltonian cycle in *G*. A survivable path is *incident to a vertex* if that vertex is one of the end-vertices of that survivable path.

Let $H_G$ be a graph derived from $G$ as follows. Graph $H_G$ consists of the same set of vertices as graph $G$, and in addition consists of edges between pairs of vertices corresponding to pairs of end-vertices of survivable paths in *G*. Define *S(G)* to be a subgraph of *G* consisting of all edges of an optimal set of survivable paths connecting end-vertices of survivable paths in *G*. Let *G(V,E)* be a graph on *V(G)* vertices and *E(G)* edges. Let $\overline{S}(G)$ be a complement of *S(G)* in respect to $G$, (i.e., $V(\overline{S}(G)) = V(S(G))$, each edge $e \in E(\overline{S}(G))$ is such that $e \notin E(S(G))$,and $e \in E(G))$. If $\overline{S}(G)$ is connected then let $\overline{T}(G)$ be a spanning tree of $\overline{S}(G)$.

## 2 Properties of Optimal Sets of Survivable Paths

Based on the above definitions, we present the following fundamental properties of the optimal sets of survivable paths, which have been proven in [2]. In particular, Theorem 2.1 represents the main result of this work.

**Theorem 2.1** *Let G be a 2-edge-connected graph with m survivable paths defined between k (m≥k) pairs of adjacent vertices. Then there exists an optimal set of survivable paths that contains k distinct and shortest working paths.*

The following corollary follows directly from Theorem 2.1:

**Corollary 2.2** *Let G be a 2-edge-connected graph with every survivable path defined between adjacent vertices and at most one survivable path defined for any pair of vertices. Then there exists an optimal set of survivable paths that contains every shortest working path.*

**Theorem 2.3** *Let G be a 2-edge-connected graph having pairwise diverse shortest working paths in an optimal set of survivable paths. Let* $\overline{S}(G)$ *be a connected spanning subgraph of G induced by the pairwise diverse shortest working paths and let* $H_G$ *be a connected graph . Then there exists an optimal set of survivable paths consisting of pairwise diverse shortest working paths and every protection path lying on a single spanning tree* $\overline{T}(G)$.

Note, Theorems 2.1, 2.3, and Corollary 2.2 hold for 2-connected graphs as well.

**Theorem 2.4** *Let G be a Hamiltonian graph with at most one survivable path defined over any pair of adjacent vertices. Let* $\overline{S}(G)$ *be a disconnected graph, and let* $H_G$ *be a connected graph. Then there exists an optimal set of survivable paths consisting of every shortest working path and every protection path lying on a single Hamiltonian cycle C(G).*

The following Corollary follows from Theorem 2.4:

**Corollary 2.5** *Let G be a Hamiltonian graph with at most one survivable path defined over adjacent pairs of vertices , and let* $H_G$ *be a connected graph. Then there exists an optimal set of survivable paths consisting of pairwise diverse shortest working paths and every protection path lying either on a single spanning tree* $\overline{T}(G)$ *or on a single Hamiltonian cycle C(G).*

**Theorem 2.6** *Let G be a Hamiltonian graph having pairwise diverse shortest paths with a distance of* $\leq 2$ *corresponding to survivable paths defined in G, and let* $H_G$ *be a connected graph . Then there exists an optimal set of survivable* *paths consisting of pairwise diverse shortest working paths and every protection path lying either on a single spanning tree of G or on a single Hamiltonian cycle C(G).*

To present Theorem 2.7 we introduce additional definitions. Let $C_1, C_2, .., C_i$ be connected components of $H_K$. Let $P_i$ be a spanning tree of $\overline{C}_i$ if $\overline{C}_i$ is connected, or a Hamiltonian cycle on vertices $V(C_i)$ if $\overline{C}_i$ is disconnected.

**Theorem 2.7** *Let $K_n$ be a complete graph of order $n > 2$ with $k$ survivable paths defined between $k$ distinct pairs of vertices. Let $C_1, C_2, .., C_i$ be connected components of $H_K$. Then an optimal set of survivable paths consists of every shortest working path and all protection paths lying on $P_1, P_2, .., P_i$.*

The following Corollary follows from Theorem 2.7:

**Corollary 2.8** *Let $K$ be a complete graph of order $> 2$ with at most one survivable path defined for any pair of adjacent vertices , and let $H_K$ be a graph consisting of components that are either triangles, squares, or a combination of both. Then there exists an optimal set of survivable paths in which the protection paths cover exactly the same set of edges as do the working paths.*

It's easy to observe that the above condition (i.e., $H_K$ being a graph consisting of either triangles or squares) is also a necessary condition for this interesting property. The cost of the solution for $m$ survivable paths in this case equals *2m*.

**Theorem 2.9** *Let $K_n$ be a complete graph of order $n$ ($n > 2$) with at most two survivable paths defined for any pair of vertices, and at most n-1 survivable paths incident to any vertex. Let $\overline{S}(K)$ be a disconnected graph, and let $H_K$ be a connected graph. Then there exists an optimal set of survivable paths consisting of every shortest working path and every protection path lying on a single Hamiltonian cycle C(K).*

Finally, we give two conditions for a graph to be 2-optimal.

**Theorem 2.10** *Let G=($V_1, V_2$,E) be a complete bipartite graph and let $|V_1|$,$|V_2|$ $\geq$ 2. Then G is 2-optimal.*

**Theorem 2.11** *Let G=(V,E) be a 2-connected graph of order at least 7. Then G is 2-optimal only if it contains a square.*

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