On Properties of Working Paths in Optimal Set of Survivable Paths

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Abstract

A *survivable path* (W, P) between a pair of vertices x_i, x_j in undirected simple graph G is an ordered pair of edge-disjoint simple paths consisting of a working path $W = x_i...x_j$ and a protection path $P = x_i...x_j$. An optimal set of survivable paths in graph G corresponds to a set of mesh-restored lightpaths defined on an optical network that minimizes the number of used optical channels. In this paper we present new properties of the working paths, which are contained in an optimal set of survivable paths in G.

Keywords: Survivable paths, Optimal set, Shortest paths, Routing.

1 Introduction

A *survivable path* (W, P) between a pair of vertices x_i, x_j in undirected simple graph G is an ordered pair of edge-disjoint simple paths consisting of a working path $W = x_i ... x_j$ and a protection path $P = x_i ... x_j$. Figure 1 illustrates a survivable path $(W, P) = (x_1x_2x_3x_6, x_1x_4x_5x_6)$ in G.

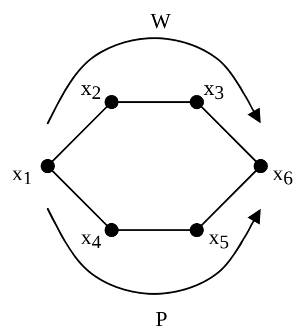

Figure 1: Survivable path between vertices x_1, x_6

A survivable path in G models a *mesh-restored lightpath* in the optical network, which has a working route and a diversely routed backup route [3-4,7]. Each edge in G consists of an infinite number of channels. A working path uses one dedicated channel on every edge of its path. A protection path also uses one channel on every edge of its path, but in addition it can share channels in such a manner as to provide guaranteed recovery upon the failure of any single edge [1]. That is, a protection path can share a channel with other protection paths if the corresponding working paths are pairwise edge-disjoint. An ***optimal set of survivable paths*** minimizes the total number of used channels over all sets of survivable paths defined between identical pairs of vertices in G. The properties of optimal set of survival paths were initially studied in [2]. The problem of finding an optimal set of survivable paths corresponds to the variation of optimal routing in a telecommunication network, which is *NP-Complete* [3-6].

Formally the optimal set of survivable paths can be defined as follows. Let $c(S)$ denote the total number of used channels by the set of survivable paths S in G. Let $(W, P) = (x_i(k)...x_j(k), x_i(k)...x_j(k))$ be k'th survivable path in S, with working path $W = x_i...x_j$ and edge-disjoint protection path $P = x_i...x_j$ between vertices x_i and x_j. Let S' be any other set of survivable paths in G (i.e., $S' \neq S$), such that if $(W', P') = (x'_i(k)...x'_j(k), x'_i(k)...x'_j(k)) \in S'$ then $x'_i(k) = x_i(k)$, and $x'_j(k) = x_j(k)$. If for every such S' $c(S) \leq c(S')$ then S is optimal.

Based on the above definition, the properties of working and protection paths in an optimal set of survivable paths were identified in [2]. In this paper, we focus on the working paths only. In Section 2 we generalize one key property from [2] and present several new properties for the working paths based on a 2-edge connected graph. In addition, in Section 3 we present several new properties of the working paths based on a complete graph.

2 Survivable Paths Defined in 2-edge-connected Graph

We call two survivable paths *distinct* if they are not defined between the same pair of end-vertices. Figure 2 illustrates two distinct survivable paths defined on G with four vertices - first one defined between end-vertices x_1, x_3, and second one defined between end-vertices x_1, x_4. Working paths are denoted by thick arrows, and protection paths are denoted by thin arrows. So, first survivable path equals $(W_1, P_1) = (x_1(1)x_3(1), x_1(1)x_2(1)x_3(1))$, and second survivable path equals $(W_2, P_2) = (x_1(2)x_4(2), x_1(2)x_2(2)x_4(2))$ in Figure 2. Note, both survivable paths here can share a channel on edge x_1x_2.

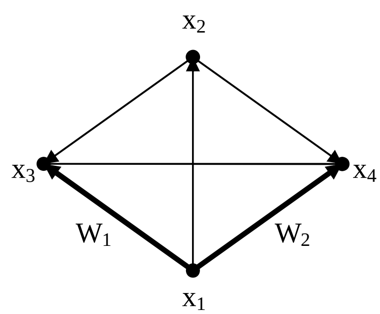

Figure 2: Example of two distinct survivable paths

For convenience we introduce the following notation for paths. Path $x_ix_{i+1}...x_j$ denotes a path at least two in length between vertrices x_i and x_j. In addition, if path x_ix_j is allowed between vertices x_i and x_j then path $x_i...x_j$ denotes a path at least one in length between them. The following is known.

Theorem 2.1 [2] *Let G be a 2-edge-connected graph with every survivable path defined between adjacent vertices and at most one survivable path defined for any pair of vertices. Then there exists an optimal set of survivable paths that contains every shortest working path.*

Our first result generalizes Theorem 2.1 as follows.

Theorem 2.2 *Let G be a 2-edge-connected graph with all m survivable paths defined between k ($m \geq k$) pairs of adjacent vertices. Then there exists an optimal set of survivable paths that contains k pairwise distinct and shortest working paths.*

Proof. Consider an optimal set S of survivable paths in G. Suppose that S contains a survivable path *(W,P)* such that $W = x_a x_{a+1} ... x_b$ and $P = x_a x'_{a+1} ... x_b$, but it does not contain a survivable path with either a working or protection path of type $x_a x_b$, for some pair of vertices x_a,x_b, ($x_a \neq x_b$). Then S must also contain at least one survivable path (W^1, P^1) such that $W^1 = x_c ... x_a x_b ... x_d$ and $P^1 = x_c ... x_d$, where $x_c \neq x_d$ and P^1 shares at least one channel with P. Otherwise, we could obtain a better set of survivable paths S' by replacing W with $x_a x_b$ - a contradiction. So, we can use the following transformation of S (i.e., Transformation A, which results in $S \rightarrow S'$) that does not increase the number of used channels. We now have two cases two consider.

Case 1. P^1 contains an edge with a channel that is not shared with any other channel.
In this case we replace P^1 with $P^2 = x_c x_d$ and then swap P^2 with W^1 generating survivable path (P^2, W^1). These two steps retain the number of used channels with the assumption that protection path W^1 does not share any channels - a worst case scenario. In the next step we replace W with $x_a x_b$ and P with path $x_a ... x_c x_d ... x_b$ generating survivable path $(x_a x_b, x_a ... x_c x_d ... x_b) = (W^3, P^3)$. By replacing W with $x_a x_b$ we decrease the number of used channels by at least one. Furthermore, P^3 requires only one additional channel because it can share all its channels with other protection paths, except for a channel on $x_c x_d$. The remaining channels can be shared as follows. P^3 shares a channel on every edge of its two subpaths $x_a ... x_c$ and $x_d ... x_b$ with W^1 of survivable path (P^2, W^1). This happens because $W^3 = x_a x_b$ is edge-disjoint with P^2. Thus Case 1 transforms S into S', where S' represents an optimal set of survivable paths (i.e., S' does not use more channels than S).

Case 2. All channels of P^1 are shared with some other channels.
In this case we replace P^1 with $P^2 = x_c x_d$ that adds one channel because we do not consider P^2 to be shared - a worst case scenario. Next we swap W^1 with P^2 generating the survivable path (P^2, W^1). This second step retains the number of used channels from the first step with the assumption that protection path W^1 does not share any channels. In the third step we replace W with $x_a x_b$ and P with path $x_a ... x_c P^1 x_d ... x_b$ generating the survivable path $(x_a x_b, x_a ... x_c P^1 x_d ... x_b) = (W^3, P^3)$. By replacing W with $x_a x_b$ we decrease the number of used channels by at least one. Furthermore, P^3 does not require additional channels because it can share all its channels with other protection

paths as follows.

(1) P^3 shares a channel on every edge of its $x_a...x_c$ with W^1 of survivable path (P^2, W^1). This happens because $W^3 = x_a x_b$ is edge-disjoint with P^2.
(2) P^3 shares a channel on every edge of its P^1 with other survivable paths. This happens because W^3 is a subpath of W^1 (and the initial survivable path (W^1, P^1) shared all channels on P^1).
(3) P^3 shares a channel on every edge of its $x_d...x_b$ with W^1 of survivable path (P^2, W^1). This happens because $W^3 = x_a x_b$ is edge-disjoint with P^2.

So, Case 2 also transforms S into S', where S' represents an optimal set of survivable paths (i.e., S' does not use more channels than S). Furthermore in both cases we did not replace any working path of type $x_i x_j$ with type $x_i x_{i+1}...x_j$, but instead we generated the new distinct working path $x_a x_b$. Consequently, Transformation A increases the number of distinct working paths of type $x_i x_j$ and does not increase the number of used channels in S'.

Consider now S such that for some adjacent vertices x_a, x_b survivable path $(W, P) = (x_a x_{a+1}...x_b, x_a x_b)$ exists, but there is no working path $x_a x_b$. We can use the following transformation of S (i.e., Transformation B, which results in $S \rightarrow S$") that does not increase the number of used channels. If no other protection path in G shares channel on edge (x_a, x_b) with P then by swapping W with P we obtain S" with no more used channels. Consider then P that shares a channel on (x_a, x_b) with at least one other protection path. In this case we first swap W with P that produces a feasible solution containing working path $W' = x_a x_b$ and new protection path $P' = x_a x_{a+1}...x_b$. Such a swap might increase the number of used channels by at most one. That is, path P' in the worst case will require exactly the same number of channels as original path W, and path W' will add one used channel. We now substitute every protection path of form $P_{ij} = x_i...x_a x_b...x_j$ that originally shared channel on edge (x_a, x_b) with P, with protection path $P'_{ij} = x_i...x_a x_{a+1}...x_b...x_j$ creating (W_{ij}, P'_{ij}), where subpath $x_a x_{a+1}...x_b = W$. We observe that P'_{ij} cannot violate diversity with W_{ij}, because otherwise P_{ij} couldn't be shared with P initially. Since we divert all such shared paths then the number of used channels decreases by one (i.e., channel on $x_a x_b$ will no longer be needed). Although P'_{ij} might contain cycle(s), but it doesn't violate either channel sharing or diversity with working path W_{ij}. In addition, it can be easily transformed to a simple protection path by $P'_{ij} = x_i...x_r...x_r...x_j \rightarrow P"_{ij} = x_i...x_r...x_j$, that preserves the channel sharing as well as diversity with W_{ij}. So, Transformation B transforms S into S", where S" represents an optimal set of survivable paths (i.e., S" does not use more channels than S). Furthermore, we did not replace any working path of type x_ix_j with type $x_ix_{i+1}...x_j$, but instead we generated new distinct working path x_ax_b. Consequently, Transformation B increases the number of distinct working paths of type x_ix_j and does not increase the number of used channels in S".

Suppose now that S contains a survivable path (W, P) such that $W = x_ax_{a+1}...x_b$, but it does not contain a survivable path with a working path of type x_ax_b. Then we can apply either Transformation A or B. By using Transformation A we generate an optimal set of survivable paths S' with more distinct working paths of type x_ix_j. By using Transformation B we generate an optimal set of survivable paths S" also with more distinct working paths of type x_ix_j. Hence, by induction there exists an optimal set of survivable paths S^* with k distinct working paths of type x_ix_j. □

Note, based on the proof of Theorem 2.2 establishing distinct direct working paths for adjacent vertices does not affect blocking of the other survivable paths. So, Theorem 2.2 has the same applicability in case of a finite number of channels on the edges.

A *1-shortest* path between two vertices x_i and x_j in G is defined as a shortest path between x_i and x_j. Furthermore, let *k+1-shortest* path between x_i and x_j be a shortest path of length greater than *k-shortest* path between x_i and x_j. Then we can present the following results.

Theorem 2.3 *Let G be a 2-edge-connected graph with m survivable paths defined between pairs of adjacent vertices. Then there exists an optimal set of survivable paths that contains m working paths, each of either type x_ix_j or 2-shortest working path.*

Proof: Suppose that an optimal set of survivable paths S contains a *k-shortest* working path W between two adjacent end-vertices x_i, x_j in G, where $k > 2$. Let k' be the length of W. Then, we can replace (W, P) with (x_ix_j, P'), where P' represents *2-shortest* protection path. The number of channels used by (x_ix_j, P') is not greater than the number of channels used by W. This is because the number of channels used by (x_ix_j, P') is at most $1 + (k' - 1) = k'$ (i.e., one channel associated with x_ix_j, and by definition at most $k' - 1$ channels associated with P'), and by definition a *k-shortest* working path W uses k' channels, where $k' \geq k \geq 3$ is satisfied. So, we transform $S \rightarrow S'$, where S' is also optimal but with fewer *k-shortest* working paths for $k > 2$. Since there are m survivable paths defined for pairs of adjacent vertices in G, then by induction there exists an optimal set of survivable paths S^* consisting of m_1 *1-shortest* and m_2 *2-shortest* working paths, where $m_1 + m_2 = m$. □

Theorem 2.4 *Let S be an optimal set of survivable paths in a 2-edge-connected* *graph G. Let k, q be given integers, where $k \geq 3, q \geq 3$. If S contains survivable path (W,P) between two adjacent end-vertices v_i and v_j in G such that W is a k-shortest path then S does not contain other survivable path (W',P') such that W' is a q-shortest path between vertices v_i and v_j.*

Proof: Suppose S contains W, W' such that W is a $k-shortest$ working path and W' is a $q-shortest$ working path between two adjacent end-vertices v_i and v_j in G, for $k \geq 3, q \geq 3$. Then, we can replace (W, P) with $(x_i x_j, P^2)$, where P^2 represents *2-shortest* protection path. The number of channels used by $(x_i x_j, P^2)$ is not greater than the number of channels used by W. Furthermore, P^2 cannot share a channel with any other protection path. Otherwise, the number of used channels by $(x_i x_j, P^2)$ would be less than the number of used channels by W - a contradiction. So, we could subsequently replace (W', P') with (W^2, P^3), where $P^3 = P^2$ and W^3 is $2-shortest$ path. So, P^2 and P^3 would share all their channels. This in turn would mean that the number of used channels would decrease - a contradiction. □

Theorem 2.5 *Let S be an optimal set of survivable paths in a 2-edge-connected graph G. Let k be given integer, where $k \geq 3$. If S contains survivable path (W,P) between two adjacent end-vertices v_i and v_j in G such that W is a k-shortest path then W is a 3-shortest path.*

Proof: Suppose W is $k-shortest$ path, for $k \geq 4$. Then, we can replace (W, P) with $(x_i x_j, P')$, where P' represents *2-shortest* protection path. The number of channels used by $(x_i x_j, P')$ is now less than the number of channels used by W - a contradiction. □

Note, Theorems 2.3-2.5 do not require all survivable paths to be defined between adjacent end-vertices in G, so in that sense it's generic. This in turn has real implication on establishing/routing the lightpaths in the optical networks [7]. In particular, if the optical network is lightly loaded and the blocking of the lightpaths is not an issue, we can establish a priori direct working lightpath routes between adjacent nodes based on Theorems 2.3-2.5. Based on the above results we also give the following statement.

`Conjecture` *Let G be a 2-edge-connected graph with a subset of m survivable paths defined for k (m≥k) pairs of adjacent vertices. Let k'≥k. Then there exists an optimal set of survivable paths that contains k' working paths of type $x_i x_j$ and remaining m-k' paths being 2-shortest working paths.*

Finally, we note that the above conjecture if proven true would have significant importance in the real-world network design and optimization. It would allow to establish a priori direct distinct working routes for the lightpaths between adjacent nodes for arbitrary set of lightpath demands. This in turn could save a critical computational time during the optical network design and optimization.

3 Survivable Paths Defined in Complete Graph

Based on the results from Section 2 we now identify the properties of the working paths in an optimal set of survivable paths in complete graph K.

Corollary 3.1 *Let K be a complete graph. Then there exists an optimal set of survivable paths defined in K that consists only from 1-shortest and 2-shortest working paths.*

Proof: Follows directly from Theorem 2.3. □

Note, *1-shortest* path in K is of length 1, and *2-shortest* path in K is of length 2. Consequently, we have the following.

Theorem 3.2 *Let S be an optimal set of survivable paths defined between m pairs of vertices in K. Then the total number of used channels $C(S)$ by working paths in S is $C(S) \leq 2|S| + m$.*

Proof: By Theorems 2.4 and 2.5 there is at most one working path of length 3 for any distinct pair of vertices in K. This contributes $3n$ to $C(S)$, $m \geq n \geq 0$. By Corollary 3.1, all remaining working paths $|S| - n$ are of length at most 2, which contribute at most $2(|S| - n)$ to $C(S)$. So, $C(S) \leq 3n + 2|S| - 2n = 2|S| + n \leq 2|S| + m$. □

Theorem 3.3 *Let S be an optimal set of survivable paths defined between m pairs of vertices in K. Then there exists an optimal set S' of survivable paths with identical end-vertices in K, where $|S| = |S'|$ and the total number of used channels $C(S')$ by working paths in S' is $C(S') \leq 2|S'| - m$.*

Proof: By Theorem 2.2 there is S', which contains at least one working path of length 1 for any distinct pair of vertices in K. This contributes m to $C(S')$, for m distinct working paths. By Corollary 3.1, all remaining working paths are of length at most 2, which contribute at most $2(|S'| - m)$ to $C(S')$. So, $C(S') \leq m + 2|S'| - 2m = 2|S'| - m$. □

4 Conclusion

In this paper we presented several new properties of the working paths, which are contained in an optimal set of survivable paths in G. These properties were defined for the edges in G with the infinite capacities in terms of the number of channels. In particular, the new property corresponding to Theorem 2.2 would have the same applicability in case of a finite number of channels on the edges, because establishing direct working paths for adjacent vertices does not affect blocking of the other survivable paths. The other properties are also applicable in this case, if G is lightly loaded and the blocking of the survivable paths is not an issue. Therefore, these new properties should be attractive for the real-world optical network design problems, where the subset of direct working paths can be established in advance, saving the computational time of the optimizations.

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