Bayesian estimation of dynamic systems function expansions

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Abstract— **Orthonormal function expansions have been used extensively in the context of linear and nonlinear systems identification, since they result in a significant reduction in the number of required free parameters. In particular, Laguerre basis expansions have been used in the context of biological/physiological systems identification, due to the exponential decaying characteristics of the Laguerre orthonormal basis, the rate of which is determined by the Laguerre parameter α. A critical aspect of the Laguerre expansion technique is the selection of the model structural parameters, i.e., polynomial model order for nonlinear systems, number of Laguerre functions and value of the Laguerre parameter α. This selection is typically made by trial-and-error procedures on the basis of the model prediction error. In the present paper, we formulate the Laguerre expansion technique in a Bayesian framework. Based on this formulation, we derive analytically the posterior distribution of the α parameter and the model evidence, in order to perform model order selection. We also demonstrate the performance of the proposed method by simulated examples and compare it to alternative statistical criteria for model order selection.**

I. INTRODUCTION

The study of many nonlinear systems has been pursued in the context of Volterra-Wiener models, which apply to a very broad class of systems; for example, these models have been extensively used to model physiological systems [1], [2]. The Volterra-Wiener framework yields a rigorous description of the system nonlinearities in the form of a hierarchy of Volterra or Wiener kernel functions, which are valid over the entire bandwidth and dynamic range of system operation. Among several approaches suggested for the estimation of these kernel functions from input-output data [1], [2], [3], [4], an approach that results in a significant reduction in the number of free parameters is to use function expansions in terms of an orthonormal basis, first suggested by Wiener [5]. For example, the Laguerre [6] and Kautz [7] bases have been employed for linear systems identification. In the case of nonlinear systems, expansion of the Volterra kernels in terms of the Laguerre basis has also proven successful [8], [9], [10]. An efficient way to estimate the Laguerre expansion coefficients utilizing least-squares estimation in connection with discrete-time Laguerre expansions [9]. The combination of Laguerre expansions with feedforward networks with polynomial activation functions was proposed in [10].

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The performance of the Laguerre expansion technique has been shown to be excellent, as long as the Laguerre parameter α and the number of basis functions are selected properly. The rate of convergence for truncated Laguerre series depends on the selection of α - it was suggested that the latter should be selected according to the dominant time constant of the system [6]. In many applications to date of Laguerre expansions, the selection of the model structural parameters, i.e., the value of α, the number of Laguerre functions L and model order Q in the case of nonlinear systems, has been empirical (based on trial-and-error procedures). These procedures are typically based on least-squares based cost functions and may also require prior assumptions about some of the system characteristics (e.g., system memory) or the use of out-of-sample data, which poses limitations for short input/output data sets. However, some theoretical work has been published on the determination of the optimal pole position of Laguerre filters for linear systems [11], [12]. In the case of nonlinear systems, a closed-form optimal value for the Laguerre poles of the expansion of a given set of Volterra kernels was derived in [13]. Finally, an efficient computational approach was introduced in order to train the Laguerre parameter α by gradient descent on the basis of the input-output data [10].

In the present paper, we formulate the Laguerre expansion technique for nonlinear systems in a Bayesian framework and we analytically calculate the posterior distribution of α, as well as the Bayesian model evidence, allowing us to infer on the values of α, L and Q on the basis of the input-output observations. We present the performance of this approach in various simulation scenarios.

II. METHODS

A. Laguerre expansion of Volterra kernels

The general Volterra model for a Q-th order nonlinear system is given below in discrete time [2]:

$$\begin{aligned} y(n) &= \sum_{q=0}^{Q} \sum_{m_1,\ldots,m_q} k_q(m_1,\ldots,m_q)x(n-m_1)\ldots x(n-m_q) \\ &= k_0 + \sum_{m} k_1(m)x(n-m) \\ &\quad + \sum_{m_1}\sum_{m_2} k_2(m_1,m_2)x(n-m_1)x(n-m_2) + \ldots \end{aligned} \quad , \tag{1}$$

where $x(n)$ and $y(n)$ are the system input and output respectively and k_q denotes the q-th order Volterra kernel of the system. The Volterra kernels describe the linear ($q = 1$) and nonlinear ($q > 1$) dynamic effects of the input on the output. The sum of eq. (1) can be viewed as a generalization

of the convolution sum, with the Volterra kernels quantifying the effect of past input values (linear kernel), as well as their q-th order products (nonlinear kernels) on the output at present time n. For causal and finite memory systems the sums in (1) are defined from $m_i = 0$ to M, where M is the system memory.

An efficient method to obtain estimates of the Volterra kernels k_q using input-output observations is to utilize function expansions of the kernels in terms of the discrete-time Laguerre orthonormal basis [9]:

$$k_q(m_1, \ldots, m_q) = \sum_{j_1=0}^{L} \cdots \sum_{j_q=j_{q-1}}^{L} c_{j_1,\ldots,j_q} b_{j_1}(m_1) \ldots b_{j_q}(m_q). \tag{2}$$

where $b_j(m)$ is the j-th order discrete-time Laguerre function given by:

$$b_j(m) = \alpha^{(m-j)/2}(1-\alpha)^{1/2} \sum_{k=0}^{j} (-1)^k \binom{m}{k} \binom{j}{k} \alpha^{j-k}(1-\alpha)^k \tag{3}$$

The Laguerre parameter α ($0 < \alpha < 1$) determines the exponential decay of the Laguerre functions and is critical for the efficiency and parsimony of the expansion. For example, using large α values leads to more efficient representations for systems with slow dynamics and/or large memory M.

By combining equations (1) and (2) and using matrix notation, the output can be expressed in terms of the expansion coefficients as:

$$\mathbf{y} = \mathbf{V}\mathbf{c} + \epsilon, \tag{4}$$

where $\mathbf{y}$ is the $(N \times 1)$ vector of output observations, $\mathbf{V}$ is a $(N \times d)$ matrix containing the convolution of the input with the Laguerre functions $v_j = x * b_j$, as well as all possible higher-order products $v_{j_1} v_{j_2} \ldots v_{j_Q}$ for $j_1, j_2, \ldots, j_Q = 0, ..., L$, and $\mathbf{c}$ is the $(d \times 1)$ vector of the unknown expansion coefficients. Note that the number of free parameters d is equal to $L + 1$ for $Q = 1$ and $L(L + 1)/2$ for $Q = 2$, due to the symmetry with respect to $j_1, j_2, \ldots, j_Q$.

The least-squares estimate of $\mathbf{c}$ is given by:

$$\hat{\mathbf{c}} = (\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{y}. \tag{5}$$

B. Bayesian inference for Laguerre function expansions

Bayesian inference is based on fitting a probabilistic model to a set of data, and summarising the results in terms of probabilities [14]. The main steps of the Bayesian approach are: (i) setting up a full probabilistic data generative model, (ii) conditioning on observed data, and (iii) evaluating the performance of the model. The fundamental Bayes equation relates the full posterior distribution of the model parameters $p(\Omega|\mathbf{y})$ given the observed data to the likelihood function $p(\mathbf{y}|\Omega)$ and the prior parameter PDF $p(\Omega)$ as:

$$p(\Omega|\mathbf{y}) = \frac{p(\mathbf{y}|\Omega)p(\Omega)}{p(\mathbf{y})}. \tag{6}$$

Often, we are interested in considering only one or some of the model parameters in isolation. For this, the corresponding marginal posterior distribution, which accounts for the uncertainty in the remaining parameters, needs to be calculated. The marginal posterior for a set of parameters ω, given the data $\mathbf{y}$ and the (complementary) set of the remaining parameters Ω^- is given by:

$$\begin{aligned} p(\omega|\mathbf{y}) &= \int p(\Omega|\mathbf{y}) d\Omega^- \\ &\propto \int p(\mathbf{y}|\Omega)p(\Omega) d\Omega^-. \end{aligned} \tag{7}$$

The normalising factor $p(\mathbf{y})$ in equation (6) is of particular importance. This quantity is termed model evidence and is equal to:

$$p(\mathbf{y}) = \int p(\mathbf{y}|\Omega)p(\Omega) d\Omega. \tag{8}$$

Model evidence takes into account both model accuracy and complexity, serving as a basis for Bayesian model comparison. We often write $p(\mathbf{y}|\mathcal{M})$ to denote the evidence of a model $\mathcal{M}$ given the observed data $\mathbf{y}$, and use this quantity to perform model comparison.

In the present case, we will infer on the expansion coefficients $\mathbf{c}$, the Laguerre parameter α and the noise variance σ^2, therefore our concatenated parameter vector is $\Omega = (\mathbf{c}, \alpha, \sigma^2)$. In the case of Gaussian white noise disturbances (i.e. $\epsilon \sim \mathcal{N}(0, \sigma^2)$), we can write the likelihood function as:

$$p(\mathbf{y}|\mathbf{c}, \sigma^2, \alpha) \sim \mathcal{N}(\mathbf{y}|\mathbf{V}(\alpha)\mathbf{c}, \sigma^2\mathbf{I}). \tag{9}$$

Note that the dependence of the likelihood on α is incorporated in the expression for $\mathbf{V}$. We considered non-informative priors for the expansion coefficients $\mathbf{c}$ and the Laguerre parameter α, and an inverse gamma prior with shape a and scale $b (a, b > 0)$ for the noise precision:

$$\begin{aligned} p(\mathbf{c}, \sigma^2) &\sim \text{inv}{-}\Gamma(\sigma^2; a, b) \\ p(\alpha) &\sim \mathcal{U}(0, 1). \end{aligned} \tag{10}$$

The posterior distributions for $\mathbf{c}$ and σ^2, conditioned on the value for α, are:

$$\begin{aligned} p(\mathbf{c}|\mathbf{y}, \sigma^2, \alpha) &\sim \mathcal{N}(\mathbf{c}|\hat{\mathbf{c}}, \sigma^2\hat{\Sigma}) \\ p(\sigma^2|\mathbf{y}, \alpha) &\sim \text{inv}{-}\Gamma(\sigma^2; \tfrac{N-d}{2} + a, A(\alpha)/2 + b), \end{aligned} \tag{11}$$

where $\hat{\mathbf{c}}$ and $\hat{\Sigma}$ denote the least squares estimates for the expansion coefficients and noise covariance matrix respectively:

$$\begin{aligned} \hat{\mathbf{c}} &= (\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{y} \\ \hat{\Sigma} &= (\mathbf{V}^T\mathbf{V})^{-1}, \end{aligned} \tag{12}$$

and the additional parameter appearing in the noise variance posterior is given by:

$$A(\alpha) = \mathbf{y}^T\mathbf{y} - \mathbf{y}^T\mathbf{V}\hat{\Sigma}\mathbf{V}^T\mathbf{y}. \tag{13}$$

The model complexity is specified by the number of Laguerre functions L and order Q. Let us for now consider

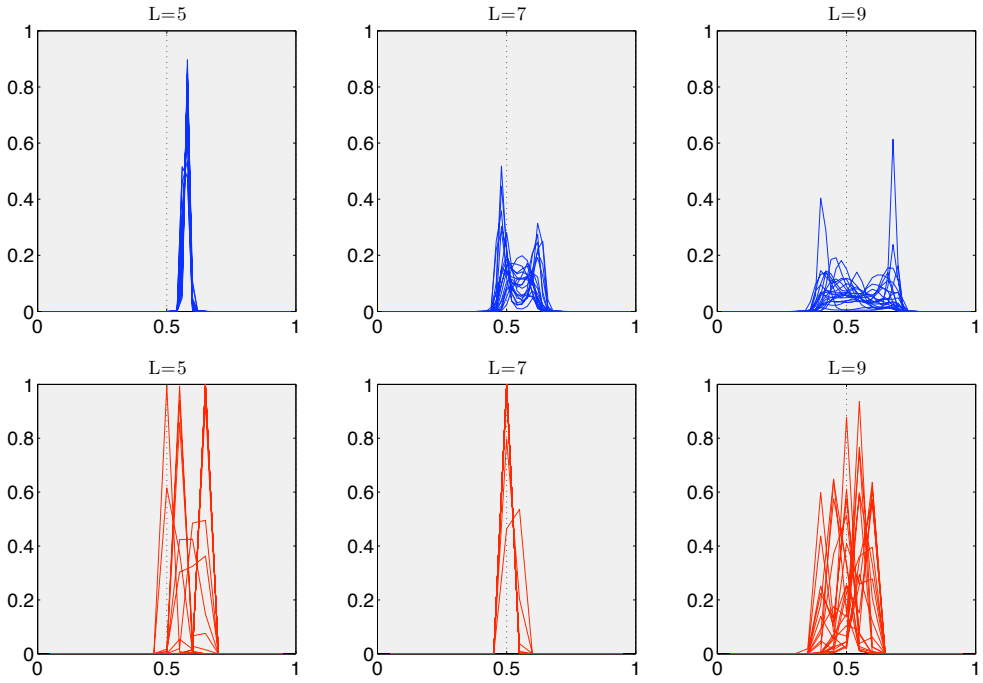

Fig. 1. Posterior distribution on α for twenty input-output realizations for SNR=10 dB in the case of linear (top) and quadratic (bottom) models.

α fixed, and denote the model by $\mathcal{M}_{LQ}(\alpha)$. The model evidence in such case writes:

$$p(\mathbf{y}|\mathcal{M}_{LQ}(\alpha)) = \iint p(\mathbf{y}|\mathbf{c},\sigma^2)p(\mathbf{c},\sigma^2)d\mathbf{c}d\sigma^2. \quad (14)$$

This integral can be calculated analytically, leading to the following formula:

$$p(\mathbf{y}|\mathcal{M}_{LQ}(\alpha)) = (2\pi)^{(d-N)/2}|\hat{\Sigma}|^{1/2}\frac{b^a}{\Gamma(a)}\frac{\Gamma(a+\frac{N-d}{2})}{(A(\alpha)+b)^{a+\frac{N-d}{2}}}. \quad (15)$$

In the above formula, the term $A(\alpha)$ accounts for model *accuracy*, while *complexity* is accounted for by the remaining factors. The quantity of (15) depends on the value of α, due to the dependence of the above equation on $\mathbf{V}$; therefore, let us denote it with $\mathcal{E}(\alpha)$. Considering the uniform prior of eq. (10) for α, we can write the marginal posterior distribution of α, which is in this case proportional to $\mathcal{E}(\alpha)$, as:

$$\begin{aligned} p(\alpha|\mathbf{y},\mathcal{M}_{LQ}) &= \frac{p(\mathbf{y}|\alpha,\mathcal{M}_{LQ})p(\alpha|\mathcal{M}_{LQ})}{p(\mathbf{y},\mathcal{M}_{LQ})} \\ &= \frac{\iint p(\mathbf{y}|\mathbf{c},\sigma^2,\alpha)p(\mathbf{c},\sigma^2)d\mathbf{c}d\sigma^2}{p(\mathbf{y}|\mathcal{M}_{LQ})} \\ &\propto \mathcal{E}(\alpha). \end{aligned} \quad (16)$$

where we have kept the conditioning on a model $\mathcal{M}_{LQ}$ with order Q and L Laguerre functions for clarity. In other words, the conditional posterior distribution on α is the previously defined model evidence in the case where α is assumed constant. Hence, given a model $\mathcal{M}_{LQ}$, one can estimate the maximum a posteriori α, i.e., as the value that maximises $\mathcal{E}(\alpha)$.

Now, having in mind that the evidence of a model $\mathcal{M}_{LQ}$ is $p(\mathbf{y}|\mathcal{M}_{LQ})$, and as the integral $\int p(\alpha|\mathbf{y},\mathcal{M}_{LQ})d\alpha$ must sum to one, this evidence is simply:

$$p(\mathbf{y}|\mathcal{M}_{LQ}) = \int \mathcal{E}(\alpha)d\alpha. \quad (17)$$

This integral has no simple analytical form. Therefore, we used numerical integration to integrate over α to compute it and compare models with different values of (L,Q).

III. RESULTS AND DISCUSSION

We considered data-generating models of the same class as the models under test, i.e., Volterra systems where the true system kernels were linear combinations of discrete-time Laguerre functions. Thus, the true model order was known. Specifically, we simulated linear and nonlinear systems using eq. (4) with $\alpha = 0.5$, $L = 7$, $Q = 1, 2$. The elements of $\mathbf{c}$ were given by values drawn from a uniform distribution between -1 and 1 and sorted in a descending order. In this manner, we obtained 25 different system realizations, keeping the same gaussian white noise input signal of length $N = 512$. The "true" system kernels k_1, k_2 are given by eq. (2) for these parameter values. In order to examine the effect of noise, we added 25 different realizations of independent white noise to the output for SNR values of 1, 10, 100 and 1000 dB; therefore, we implemented 625 different realizations for each SNR value. The posterior distributions of α, obtained from different noise realizations for one representative system realization, are shown in the top panel of Fig. 1 for SNR=10 dB and $3 \leq L \leq 9$. Note that

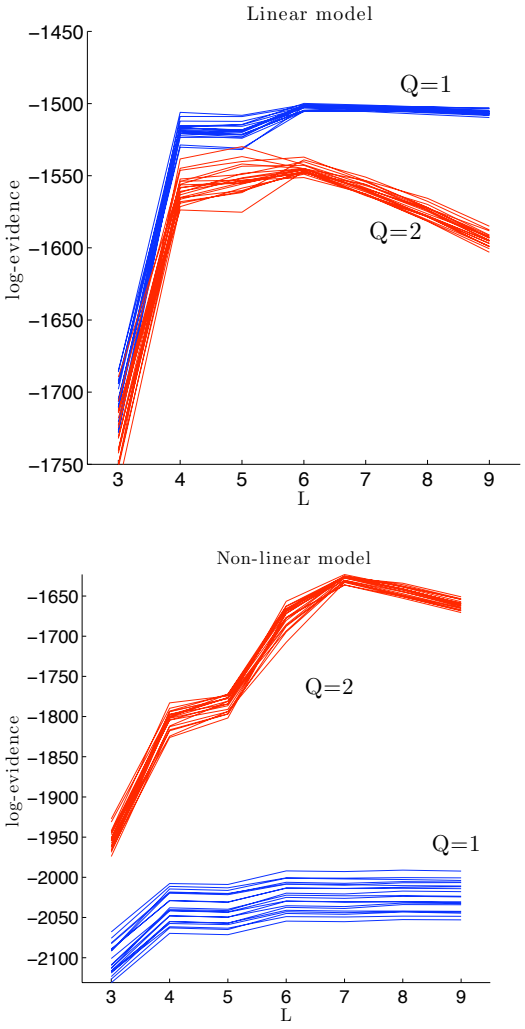

Fig. 2. Model evidence values for $Q = 1$ (blue) and $Q = 2$ (red) for 20 different realizations.

for less complex models than the true system ($L < 7$), sharp peaks in the posterior α distribution at values larger than 0.5 were observed. This reflects the fact that larger α values are favored in order to account for the slower dynamics of the higher-order basis functions that are not included in the model. On the other hand, we observed a sharp peak at or around 0.5 for $L = 7$, as well as peaks both below and above 0.5 for $L > 7$ (bimodal distributions).

The optimal model $\mathcal{M}_{LQ}$ is selected by integrating $\mathcal{E}(\alpha)$ over α and comparing its values between different models. In the case of a linear true system ($Q = 1$), the true value of Q was selected in all cases; this can be seen for a representative model realization in 2, where the values of $p(\mathbf{y}|\mathcal{M}_{LQ})$ are shown in for different input-additive noise realizations (SNR=10 dB) and $Q = 1$ and 2. With regards to the value of L, the value that was selected by the model evidence criterion (ME) is shown in Fig. 3 (left panels) for SNR=1, 10 and 100 dB, along with the corresponding results obtained when using the Akaike and Bayesian information

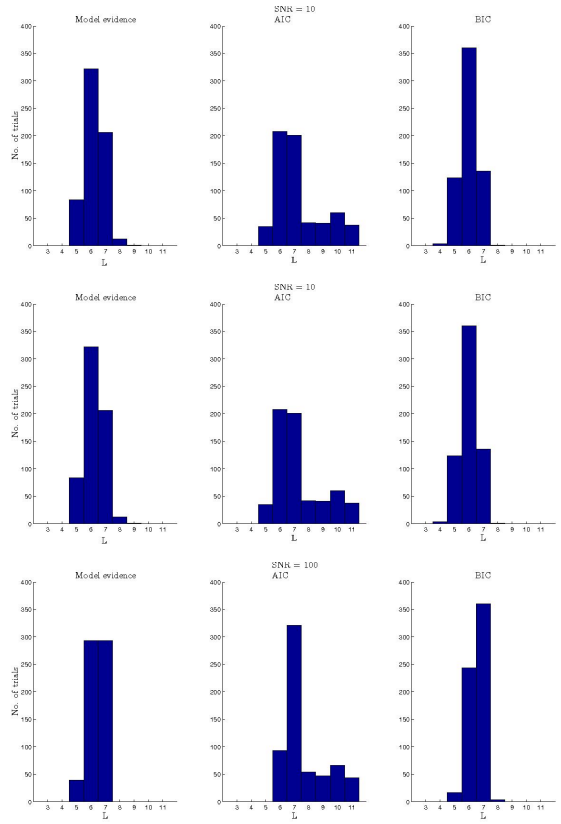

Fig. 3. Optimal L values selected by model evidence, Akaike and Bayesian information criteria for a true system with $Q = 1$, for a total of 625 different realizations and SNRs of 1, 10 and 100 dB.

criteria (AIC and BIC - middle and right panels respectively). Overall, the ME criterion was affected less by the presence of noise; the obtained distributions for the selected L did no exhibit any clear bias for low SNRs. On the other hand, the AIC tended to overestimate model complexity (there was a bias towards values $L > 7$) even for high SNRs, while the BIC tended to underestimate model complexity for low SNRs (Fig. 3). For a true nonlinear system with $Q = 2$, the resulting posterior α distributions and model evidence values are shown in the bottom panels of fig. 1 and 2 respectively, for SNR=10 dB.

IV. CONCLUSION

We formulated Laguerre expansions of Volterra kernels in a Bayesian framework and used the results to infer on the expansion structural parameters, with the results suggesting better overall performance than the AIC and BIC. The calculation of uncertainty bounds for the expansion coefficient and Volterra kernel estimates, as well as the evaluation of the proposed approach using physiological data are currently underway. Also, an issue that deserves further attention is ill-conditioning of the Gram matrix $\mathbf{V}^T\mathbf{V}$, which occurs for large α values, short input/output records and/or low SNRs, which may bias the α posterior distribution.

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