Joint Preprocessing and Feedback strategies for Perfectly Reconstructing Equalizers*

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Abstract

In this paper we consider the transmission of discrete-valued data via a communication channel that is subject to (additive) noise with a known upper bound on its magnitude but otherwise completely unrestricted and unknown behavior. We extend previous equalization strategies for perfect reconstruction by allowing preprocessing of the data and/or linear feedback from the receiver to the transmitter. We are interested in the characterization of general conditions that allow perfect reconstruction of the discrete data (with any given delay and under all possible realizations of channel noise and a limit on the power of transmission) when linear preprocessing of the data and/or linear feedback from the receiver is employed. In particular, we obtain necessary and sufficient conditions for perfect reconstruction under linear power-limited preprocessing and under linear power-limited preprocessing along with linear feedback. We also consider the case when a Decision Feedback Equalizer (DFE) structure is imposed at the receiver and provide necessary conditions for improvements in the perfect reconstruction in terms of $\ell^1$ norms of appropriate maps. In particular, we prove that a necessary condition for improving the perfect reconstructability margin is instability of the feedback and preprocessing systems. In addition, a procedure that results in parametric $\ell^1$ optimization is developed to design a DFE to improve maximum tolerable noise bound.

Keywords: Discrete-valued signal reconstruction, $\ell^1$ optimality, worst-case, equalization

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1 Introduction and Motivating Applications

The study of data transmission and reconstruction has been based almost entirely on stochastic formulations of the various problems involved (e.g., [1, 2]). In these formulations, the measure of performance for a communication system is characterized primarily in terms of the probability of error under stochastic assumptions on the noise and channel behavior. Designing a system that minimizes this probability is a hard problem and the proposed algorithms are characterized by high complexity (e.g., Viterbi’s algorithm [1]). In our earlier work in [3, 4] we presented a deterministic worst-case framework for perfect reconstruction of discrete (source) data transmissions. The particular problems touched upon included: (i) necessary and sufficient conditions for causal (without delay) and non-causal (with delay) perfect reconstruction under deterministic, magnitude-bounded noise for single-input single-output (SISO) and multi-input multi-output (MIMO) channels; (ii) perfect reconstruction based on decision feedback equalizer (DFE) and linear structures; (iii) perfect reconstruction under channel uncertainty. As shown by our developments in [3], our worst-case deterministic approach to perfect reconstruction leads to novel and attractive designs of DFEs or linear equalizers.

There are a number of applications where unknown but bounded noise models can be more realistic than additive white Gaussian noise (AWGN) channels. For example, recent studies on modeling of high speed links in chip-to-chip or board-to-board communication (that consider CMOS components to generate, receive and recover timing of high-speed data [5]) show that the dominant noise sources are colored and bounded. Actual designs and measurements for such non-standard scenarios are also provided in [6, 7]. Furthermore, there are applications where quantization is a dominant noise source and, as such, it is of a bounded non-AWGN type. This type of applications can be found in signal/image processing [8] and in estimation literature [9]. Another motivation for a worst-case approach comes from applications where security to attacks by malicious agents (e.g., jammers [10]) is of paramount importance and therefore “hard” (non-probabilistic) guarantees are required. Moreover, in terms of its ability to provide “hard” guarantees, our framework is relevant to recent approaches for controlling a system over a feedback loop that is closed via a constrained communication link (see, for example, the approaches in [11, 12] and references therein).

In this paper we consider the transmission of discrete data via a communication channel that is subject to additive noise. For the noise we assume that we have no data other than a knowledge of an upper bound on its magnitude and (in the case of filtered noise) the coloring filter. We consider
the design of transmitter-receiver pairs that include preprocessing along with a rate drop as well as preprocessing along with feedback, and derive necessary and/or sufficient conditions under which we are able to perfectly reconstruct the discrete data with a given delay under all possible realizations of channel noise. As a particular choice for the receiver, we consider a decision feedback equalizer (DFE) structure while we impose a linear transmitter structure together with the requirement that the power of the transmission is limited. Under these circumstances, we provide necessary and sufficient conditions for perfect reconstruction in terms of $\ell^1$ norms of appropriate maps. We also investigate the conditions under which feedback (from the received signal to the transmitter) can help us improve the “perfect reconstructability margin,” i.e., the maximum noise bound under which we can guarantee perfect reconstruction. In particular, we show that feedback without preprocessing will not increase the perfect reconstructability margin; moreover, we argue that the necessary condition of improving the perfect reconstructability margin is that both feedback and preprocessing systems should be unstable. We would like to mention that consideration of feedback communication schemes from a combined control and information theoretic point of view have recently received considerable attention (e.g., [13, 14]).

The organization of this paper is as follows. We first review existing definitions and results that allow us to derive some theoretical results on LTI preprocessing without feedback. Then, we start the discussion on the effectiveness of feedback by presenting results on feedback without preprocessing as well as results on feedback with preprocessing. Finally, we present a design procedure and discuss relevant implementation issues for DFE receiver structures.

The notation in the paper is as follows: $\|x\|_{\infty} := \sup_k |x(k)|$ is the $\ell^\infty$ norm of the sequence $x = \{x(k)\}_{k=0}^\infty$ or the vector-valued signal $x = (x_1, x_2, \ldots, x_n)'$; $\|T\| := \sum_{k=0}^\infty |t(k)|$ is the $\ell^1$ norm of the linear time-invariant (LTI) system $T$ having unit pulse response $\{t(k)\}_{k=0}^\infty$; $\hat{T}(z) := \sum_{k=0}^\infty t(k)z^{-k}$ is the $z$-transform of $T$; for MIMO systems $T = \{T_{ij}\}$ where $T_{ij}$ are SISO, $\|T\| := \max_i \sum_j |T_{ij}|$; $T$ will be called stable if $\|T\| < \infty$; $\|T\|_{\ell^\infty} := \sup_{\|\lambda\|\leq 1} \sqrt{\rho(\hat{T}^*(1/\lambda)\hat{T}(1/\lambda))}$, where $\rho[\bullet]$ stands for the maximum modulus eigenvalue of the matrix argument and $*$ stands for complex conjugate transpose.

## 2 Preliminaries and Related Previous Work

The basic setup we are concerned with is depicted in Figure 1. Signal $s$ is a binary signal to be transmitted with $s(k) \in \{-1, 1\}$ for all $k = 0, 1, \ldots$. The noise sequence $n$ satisfies $|n(k)| \leq b$ where
Figure 1: Basic problem formulation.

$b$ is a known constant and enters the channel through a known stable LTI and stably invertible system $W$ (note that the assumption of $W$ being stably invertible is a standard assumption in the filtering literature [15]); we define $\bar{n} = \frac{b}{n}$ to be the normalized input noise sequence. System $H = \{h_0, h_1, \ldots\}$ is a stable (real coefficient) LTI system and represents the channel dynamics which are also assumed known \textit{a priori}. In previous work [3], we discussed the accurate reconstruction of $s$ via the (receiver) structure $R$.

Here we review some of the notation and results in [3] that will allow us to extend the basic setup in Figure 1 to formulations that allow preprocessing and feedback illustrated in Figure 2.

**Definition 2.1** The causal non-identical sequences $s_1 = \{s_1(k)\}_{k=0}^{\infty}$ and $s_2 = \{s_2(k)\}_{k=0}^{\infty}$ are indistinguishable at $t$ if $s_1(m) \neq s_2(m)$ for some $0 \leq m \leq t$ and there exist $n_1 = \{n_1(k)\}_{k=0}^{\infty}$, $n_2 = \{n_2(k)\}_{k=0}^{\infty}$ with $\|n_1\|_\infty \leq b$, $\|n_2\|_\infty \leq b$ such that $r_1(k) = r_2(k)$ for all $k = 0, 1, \ldots, t$, where $r_1 = Hs_1 + n_1$ and $r_2 = Hs_2 + n_2$.

Note that the above definition of indistinguishability is in agreement with the same concept in information theory [17]. Clearly, perfect reconstruction is possible causally in time (with no delay) if and only if no pair of signals $s_1$, $s_2$ are indistinguishable. Also, the necessary and sufficient condition for perfect signal reconstruction with $K$-step delay is that there are no sequences $s_1$ and $s_2$ such that they are indistinguishable at any time $t$ and they remain indistinguishable for the next $K$ time steps. Generalizations of these concepts to systems with multiple inputs and multiple outputs (MIMO) are straightforward. The following proposition gives the conditions we need to check in order to find the perfect reconstructability margin (defined below) for a known MIMO channel $H$ with $p$ inputs and $q$ outputs and a given reconstruction delay $K$ [3].

**Proposition 2.1** The necessary and sufficient condition for perfect signal reconstruction with delay
\[ K \text{ is} \]
\[
\min_{v(0) \neq 0, v(i) \in \{-1, 0, 1\}^p} \max \{ \|a(0)\|_\infty, \ldots, \|a(K)\|_\infty \} > b, \quad (1)
\]

where \( b \) is the noise bound,
\[
\begin{pmatrix}
a(0) \\
a(1) \\
\vdots \\
a(K)
\end{pmatrix} := (W^{-1}H)_K
\begin{pmatrix}
v(0) \\
v(1) \\
\vdots \\
v(K)
\end{pmatrix},
\]

and
\[
(W^{-1}H)_K = \begin{pmatrix}
W_0 & & \\
W_1 & W_0 & \\
\vdots & & \ddots \\
W_K & \ldots & \ldots & W_0
\end{pmatrix}^{-1}
\begin{pmatrix}
H_0 \\
H_1 & H_0 \\
\vdots & & \ddots \\
H_K & \ldots & \ldots & H_0
\end{pmatrix},
\]
is the convolution matrix that corresponds to the first \( K \) steps of the pulse response of the system \( W^{-1}H \).

**Definition 2.2** For a given channel \( H \), noise filter \( W \) and delay \( K \) the perfect reconstructability margin, \( B_{\text{max}}(H, W, K) \), is defined to be the largest \( b \) that satisfies Eq. (1).

Note that the above definition does not depend on a particular receiver set up. In [3], we showed that a decision feedback equalizer (DFE) is a receiver structure that (with appropriate choices) achieves the above perfect reconstructability margin for \( K = 0 \) and \( K = 1 \), but this is not necessarily true for larger delays. Also notice that the problem of checking that the noise bound satisfies Eq. (1) can be rewritten as
\[
\min_{v(0) \neq 0, v(i) \in \{-1, 0, 1\}^p} \|(W^{-1}H)_Kv\|_\infty > b. \quad (3)
\]

In this paper, we are interested in how much we can improve the perfect reconstructability margin (i.e., the maximum allowable value for \( b \)) by allowing preprocessing of the input signal \( s \) via preprocessing \( E_1 \) and possibly via feedback from the received signal \( r \) through \( E_2 \) (refer to Figure 2). In the process, we need to guarantee that the channel input signal is not amplified (i.e., the power of the transmission does not increase). Hence, we pose the condition
\[
\| \begin{pmatrix}
s \\
\tilde{n}
\end{pmatrix} \longrightarrow u \| = \|(I - E_2H)^{-1} [E_1 \ E_2Wb]\| \leq 1, \quad (4)
\]
which is equivalent to saying that we want to preserve the instantaneous power of the transmitted signal. To investigate the ultimate possible improvement in the perfect reconstructability margin, we should not impose any other constraints on $E_1$ and $E_2$, and this is the route we follow initially. However, in Section 7 where we consider practical receiver designs, we enforce additional constraints to capture the delay in the feedback path from the receiver to the transmitter.

First, we start with the case when no feedback is utilized ($E_2$ in Figure 2 satisfies $E_2 = 0$).

3 Preprocessing in the Absence of Feedback

In the case of preprocessing without feedback ($E_2 = 0$), the following proposition holds true.

**Proposition 3.1** If $W^{-1}HE_1 = E_1W^{-1}H$, then preprocessing cannot help improve $B_{\text{max}}$. In other words, a necessary condition for improving the perfect reconstructability margin is that the commutation property does not hold for preprocessor $E_1$ and $W^{-1}H$.

**Proof** We show that if the commutation property holds then preprocessing under the condition of Eq. (4) cannot improve the perfect reconstructability margin. Recall that $B_{\text{max}}$ is defined to be the maximum $b$ that satisfies Eq. (3), so we can write

$$
\begin{align*}
\max_{\|E_1\| \leq 1} \min_{v(0) \neq 0, v(i) \in \{-1,0,1\}^q} \| (W^{-1}HE_1)Kv \|_\infty \\
=^a \max_{\|E_1\| \leq 1} \min_{v(0) \neq 0, v(i) \in \{-1,0,1\}^q} \| (E_1W^{-1}H)Kv \|_\infty \\
\leq^b \max_{\|E_1\| \leq 1} \min_{v(0) \neq 0, v(i) \in \{-1,0,1\}^q} \| E_1 \| (W^{-1}H)Kv \|_\infty \\
=^c \min_{v(0) \neq 0, v(i) \in \{-1,0,1\}^q} \| (W^{-1}H)Kv \|_\infty
\end{align*}
$$
where (a) is due to the assumption $W^{-1}HE_1 = E_1W^{-1}H$, (b) is a result of sub-multiplicative inequality, and (c) is due to the condition in Eq. (4). Therefore, we can see that $B_{\text{max}}(HE_1, W, K) \leq B_{\text{max}}(H, W, K)$; clearly, since the equality holds for $E_1 = I$, $B_{\text{max}}(HE_1, W, K) = B_{\text{max}}(H, W, K)$. □

**Corollary 3.1** For SISO channels no preprocessor can improve the perfect reconstructability margin.

**Proof** For SISO systems the commutative property holds, thus we can immediately invoke Proposition 3.1 and the proof is complete. □

If one is willing to drop the rate of communication, then Corollary 3.1 is not necessarily true. In the next section, we discuss a case where preprocessing along with a rate drop in a SISO system can indeed improve the perfect reconstructability margin. In fact, we show how to choose the preprocessing optimally so as to maximize the perfect reconstructability margin.

### 4 Optimal Preprocessing under a Power Constraint

In this section, we consider the setup in Figure 2 with $E_2 = 0$ (i.e., with no feedback). We are interested in understanding how a drop in the rate of communication (i.e., a reduction in the number of symbols transmitted per time step) can help improve the maximum noise bound that is required for perfect reconstruction. To keep the notation simple we fix $W = I$ but the analysis can easily be adjusted for the case when the noise is colored. Specifically, we consider the setup in Figure 4 where the processor $M$ is a single-input $D$-output LTI system $M = \{M_0, M_1, \ldots\}$ that maps the binary input sequence $s$ to an $\mathbb{R}^D$-valued sequence $\bar{u}$ as $\bar{u} = Ms$. The output

$$
\bar{u} = \begin{bmatrix}
\bar{u}_1(0) \\
\vdots \\
\bar{u}_D(0)
\end{bmatrix},
\begin{bmatrix}
\bar{u}_1(1) \\
\vdots \\
\bar{u}_D(1)
\end{bmatrix}
$$

gets interleaved to produce a single $\mathbb{R}$-valued sequence as

$$
u = \{\bar{u}_1(0), \ldots, \bar{u}_D(0), \bar{u}_1(1), \ldots, \bar{u}_D(1), \ldots\} = L\bar{u},
$$

where the interleaving operator is denoted by $L$. Sequence $u$ is fed to $H$ and the result is corrupted by the additive noise $n$ to produce the received sequence

$$
r = \{r(0), \ldots, r(D - 1), r(D), \ldots, r(2D - 1), \ldots\},
$$
which is subsequently de-interleaved via the operator \( L^{-1} \)
\[
\tilde{r} = \left\{ \left( \begin{array}{c} r(0) \\ \vdots \\ r(D-1) \end{array} \right), \left( \begin{array}{c} r(D) \\ \vdots \\ r(2D-1) \end{array} \right), \ldots \right\} = L^{-1}r.
\]

Figure 3: Channel with data preprocessing.

Based on \( \tilde{r} \), the receiver \( R \) reconstructs \( s \) by producing \( \hat{s} \). Note that in order to produce \( \hat{s}(k) \) the scheme requires information from \( \{\tilde{r}(0), \ldots, \tilde{r}(k)\} \) i.e.,
\[
\left\{ \left( \begin{array}{c} r(0) \\ \vdots \\ r(D-1) \end{array} \right), \left( \begin{array}{c} r(D) \\ \vdots \\ r(2D-1) \end{array} \right), \ldots, \left( \begin{array}{c} r(kD) \\ \vdots \\ r((k+1)D-1) \end{array} \right) \right\}.
\]

Essentially, the preprocessing scheme of Figure 4 drops the rate of transmission by \( D \). The above set-up can be equivalently seen as a single-input \( D \)-output channel as in Figure 4 with \( \tilde{H} = \)

8
\[ L^{-1}HLM = \{ \tilde{H}_0, \tilde{H}_1, \cdots \}. \] If \( M_0 = \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{D-1} \end{pmatrix} \), then

\[ \tilde{H}_0 = \begin{pmatrix} h_0 & & & \\ h_1 & h_0 & & \\ & \ddots & \ddots & \ddots \\ h_{D-1} & \cdots & \cdots & h_0 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{D-1} \end{pmatrix}. \]

The (equivalent) noise \( n' \) is bounded as \( \|n'\| \leq b \). Thus, based on the MIMO condition for perfect reconstruction in Proposition 2.1 (with \( K = 0 \)), we have that \( R \) can perfectly reconstruct \( s \) if and only if \( \|\tilde{H}_0\| > b \) for the equivalent system.

We now consider the problem of optimally designing \( M \) without increasing the per-symbol energy. This constraint translates to \( \|M\|_{\mathcal{H}_\infty} \leq 1 \). As the only part of \( M \) that enters \( \tilde{H}_0 \) is \( M_0 \), we only have the constraint \( \|M_0\|_{\mathcal{H}_\infty} \leq 1 \) (while setting \( M_i = 0 \) for \( i \geq 1 \)) or,

\[
\sum_{i=0}^{D-1} m_i^2 = m_0^2 + m_1^2 + \cdots + m_{D-1}^2 \leq 1.
\]

Thus, the problem of optimal design of a power-limited \( M \) can be formulated as maximizing

\[
\left\| \begin{pmatrix} h_0 & & & \\ \vdots & \ddots & & \\ h_{D-1} & \cdots & \cdots & h_0 \end{pmatrix} \begin{pmatrix} m_0 \\ \vdots \\ m_{D-1} \end{pmatrix} \right\|,
\]

under the constraint \( \sum_{i=0}^{D-1} m_i^2 \leq 1 \). Clearly, the problem is equivalent to

\[
\nu := \max \left\{ \sum_{i=0}^{D-1} h_{D-i-1}m_i \right\}, \text{ subject to } \sum_{i=0}^{D-1} m_i^2 \leq 1.
\]

The optimal \( m_i \) can be easily obtained as \( m_i^\circ = \frac{h_{D-i-1}}{\sqrt{\sum_{i=0}^{D-1} h_i^2}} \) and the optimal cost is \( \nu = \frac{\sum h_i^2}{\sqrt{\sum h_i^2}} = \sqrt{\sum_{i=0}^{D-1} h_i^2} \). The following proposition summarizes the developments in this subsection.

**Proposition 4.1** In the set-up of Figure 3, signal \( s \) is perfectly reconstructible at the rate of one symbol for every \( D \) transmissions while keeping the per-symbol energy constant, if and only if \( b < \sqrt{\sum_{i=0}^{D-1} h_i^2} \).
The optimal preprocessor turns out to be what is known as a matched filter [1]. One should keep in mind that this is a solution to a problem that is quite different from the one traditionally leading to matched-filter-type solutions (for instance, the problem does not involve any probabilistic characterization of noise). We should also note that one can incorporate into the problem an additional reconstructing (fixed) delay $K$ (besides the rate drop $D$) and utilize Condition (1) of Proposition 2.1 for perfect reconstruction. However, the solution to the underlying optimization is not as simple.

Note that related preprocessing (or precoding) approaches have also been studied in a stochastic setting where the goal is to mitigate the effects of intersymbol interference by essentially transforming an arbitrary fading channel into a nonfading, simple white Gaussian noise channel [18, 19] (see also [20] for an application in choosing optimal equalizers based on MMSE channel estimates and [21] for a comprehensive introduction to more general, including nonlinear, precoding techniques).

By taking a slightly modified viewpoint, one can look at the energy per symbol that is required to achieve perfect reconstruction at different transmission rates. In other words, we allow the energy per symbol to vary by transmitting binary information at $\pm A$; then, given a channel $H$ and a fixed noise level $b$, we can easily find the minimum required value for $A$ such that perfect reconstruction is possible. More specifically, with rate drop $D$, the above analysis indicates that the minimal required $A$ is

$$A = \frac{b}{\sqrt{\sum_{i=0}^{D-1} h_i^2}}.$$

In Figure 5, we show how the minimal energy-per-symbol ($A^2$) and the rate drop ($D$) are related for the case when $b = .9$ and the channel is given by

$$h = \left\{1, \frac{1}{2}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{10}, -\frac{1}{20}, 0, 0, \ldots\right\}.$$

In the next section, we discuss the effect of LTI feedback (with or without preprocessing) on the perfect reconstructability margin.

5 Feedback Effectiveness

The purpose of this section is to investigate conditions under which the perfect reconstructability margin can be improved by introducing feedback and/or preprocessing. We now assume that the transmitter receives feedback from the receiver as depicted in Figure 9. We require the signal $u$
generated by the transmitter and sent to channel $H$ to be composed as

$$u = E_1 s + E_2 r$$

where $E_1$ and $E_2$ are LTI and causal. The constraint on transmission power is captured by Eq. (4) which ensures that in the closed loop $\|u(k)\|_\infty \leq 1$ at all times $k$.

The following lemma will be used extensively throughout the paper.

**Lemma 5.1** For any causal system $X$, $B_{\text{max}}(XH, XW, K) \leq B_{\text{max}}(H, W, K)$; if $X$ is causally invertible, then $B_{\text{max}}(XH, XW, K) = B_{\text{max}}(H, W, K)$.

**Proof** First we will show that any two sequences that are indistinguishable under channel $H$, noise filter $W$ and delay $K$, they remain indistinguishable under channel $XH$, noise filter $XW$ and delay $K$. If two sequences $s_1$ and $s_2$ are indistinguishable at time $t$ under channel $H$, noise filter $W$ and delay $K$, then for some noise sequences $n_1$ and $n_2$ we have $r_1(k) = r_2(k)$ for $k = 0, \ldots, t + K$ where $r_1 = Hs_1 + n_1$ and $r_2 = Hs_2 + n_2$ (bottom diagram in Figure 6); therefore, $\hat{r}_1 = \hat{r}_2$ where $\hat{r}_1 = Xr_1$ and $\hat{r}_2 = Xr_2$ which can be seen as the responses of channel $XH$ and noise filter $XW$. This shows that, given a fixed noise bound $b$, if two sequences are indistinguishable under channel $H$ and noise filter $W$, then these sequences are indistinguishable under channel $XH$ and noise filter $XW$; thus, $B_{\text{max}}(XH, XW, K) \leq B_{\text{max}}(H, W, K)$.

If $X$ is causally invertible (i.e., if $X_0$, the first tap of its impulse response, is invertible), then
there is a one-to-one mapping between $r$ and $\hat{r}$ and indistinguishability of the system under channel $XH$ and noise filter $XW$ results in indistinguishability of the system under channel $H$ and noise filter $W$; thus, $B_{\text{max}}(XH, XW, K) = B_{\text{max}}(H, W, K)$.

Figure 6 shows the intuition behind the above lemma: introducing system $X$ is equivalent to confining the receiver $\hat{R}$ to be of the form $XR$. If the system $X$ is invertible, then for any (optimal or otherwise) receiver $\hat{R}$ in the original setup with channel $H$ and noise filter $W$, there exists an equivalent receiver in the right side of the form $XR$. However, this may not be true if $X$ is not invertible.

In the above lemma, one can observe that when $X$ is not invertible, a larger delay may allow us to achieve the same $B_{\text{max}}$ that we can achieve without $X$. As an example of this scenario, one can think of $X$ as the unit delay element ($X(z) = z^{-1}$).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6}
\caption{Illustration of Lemma 5.1.}
\end{figure}

### 5.1 Effect of Feedback

We now consider the problem of designing feedback $E_2$ along with the preprocessor $E_1$ under the “power” constraint of Eq. (4). Reconsidering Figure 2, we can write

$$r = H(I - E_2H)^{-1}E_1s + (I - HE_2)^{-1}Wn.$$  

Using the identity $H(I - E_2H)^{-1} = (I - HE_2)^{-1}H$, the problem can be rewritten as

$$r = (I - HE_2)^{-1}[HE_1s + Wn], \quad (5)$$
with the condition in Eq. (4). Now we can propose the following Theorem on the effect of feedback when preprocessing is power limited.

**Theorem 5.1** If preprocessing $E_1$ is norm bounded as $\|E_1\| \leq 1$, then there is no pair of feedback $E_2$ and preprocessing $E_1$ that can improve the perfect reconstructability margin over the perfect reconstructability margin achieved by preprocessing $E_1$ alone ($E_2 = 0$).

**Proof** The proof is by contradiction: assume that there exists a pair of feedback $E_2$ and preprocessing $E_1$ (with $\|E_1\| \leq 1$) that achieve better perfect reconstructability margin than preprocessing $E_1$ alone while the condition in Eq. (4) is satisfied, i.e.,

$$B_{\text{max}}((I - HE_2)^{-1}HE_1, (I - HE_2)^{-1}W, K) > B_{\text{max}}(HE_1, W, K).$$

Then, if we set $HE_1$ and $(I - HE_2)^{-1}$ to $H$ and $X$ respectively in Lemma 5.1, we reach a contradiction.

The above theorem can also be expressed as follows.

**Corollary 5.1** A necessary condition for improving the perfect reconstructability margin by using feedback $E_2$ and preprocessing $E_1$ is that the pair $E_1$ and $E_2$ must satisfy Eq. (4) and $E_1$ should be either unstable or satisfy $\|E_1\| > 1$.

Note that feedback without preprocessing (i.e., $E_1 = I$), is a special case of Theorem 5.1 and can be expressed as the corollary below.

**Corollary 5.2** Regardless of the receiver structure, when no preprocessing is applied, linear feedback cannot improve the perfect reconstructability margin.

As an intermediate problem, considering Corollary 5.1, we observe that in order to improve the perfect reconstructability margin, we need to search among the set of pairs $E_1$ and $E_2$ that satisfy the following two conditions

$$\left\| \begin{pmatrix} s \\ \bar{n} \end{pmatrix} \right\| \rightarrow u \leq 1,$$

$$\|E_1\| > 1.$$

(6)

Since the above description is non-convex, it is hard to formulate it as an optimization problem that allows us to determine the pairs $E_1$ and $E_2$ that improve the perfect reconstructability margin.
5.2 Example

As a numerical example, consider the following (SISO) channel and coloring filter:

\[ H(z) = 15 - 8z^{-1} + z^{-2}, \]
\[ W(z) = 1. \]  \hspace{1cm} (7)

Using the indistinguishability criteria of Proposition 2.1 and due to the small number of taps of the channel, it is possible to use a brute-force search to obtain the exact solution to the MILP described in Eq. (3). The solution is

\[ B_{\text{max}}(H, W, K) = 15 \]

for any \( K \geq 0 \). We also find that the preprocessor and feedback pair

\[ E_1(z) = \frac{0.46z^{-1} - 0.736z^{-2} + 0.2948z^{-3}}{1 - 4.771z^{-1} + 2.275z^{-2} + 0.6538z^{-3}}, \]
\[ E_2(z) = \frac{0.8}{1 - 3.9909z^{-1} - 0.8382z^{-2}}, \]  \hspace{1cm} (8)

satisfy the condition of Eq. (4) and Theorem 5.2 while

\[ B_{\text{max}}((I - HE_2)^{-1}HE_1, (I - HE_2)^{-1}W, 2) = 16.82. \]

Note that in this example \( E_1 \) is unstable (\( \|E_1\| \) is infinite), and \( E_1 \) and \( E_2 \) increase the perfect reconstructability margin but are not necessarily the maximizers of \( B_{\text{max}}((I - HE_2)^{-1}HE_1, (I - HE_2)^{-1}W, 2) \).

\[ \begin{array}{c}
\text{\textbf{Figure 7: Formulation of Theorem 6.1.}}
\end{array} \]

6 Stability Considerations

The numerical example in Section 5.2 normally leads to concerns about the stability of the preprocessor \( E_1 \) as well as the feedback element \( E_2 \). The following theorem describes the results in this regard.
**Theorem 6.1** There is no pair of stable preprocessor \(E_1\) and feedback \(E_2\) that can increase the perfect reconstructability margin. In other words, a necessary condition for increasing the perfect reconstructability margin is instability of \(E_1\) and \(E_2\).

**Proof** The proof is by contradiction. Assume that there are stable \(E_1\) and \(E_2\) that satisfy the conditions of Theorem 5.1 (and achieve better \(B_{\text{max}}\)). Thus, we can find a finite gain \(\rho > 1\) such that the \(\ell^1\) norm of \(E'_1 := \frac{E_1}{\rho}\) is less than one. Now, we can see that Figure 7 with \(E'_2 = \frac{E_2}{\rho}\) is equivalent to our original setup in Figure 2. Since

\[
\| \begin{pmatrix} s \\ \tilde{n} \end{pmatrix} \to e \| < \| \begin{pmatrix} s \\ \tilde{n} \end{pmatrix} \to u \| \leq 1
\]  

(9)

and \(\|E'_1\| < 1\), we can invoke Theorem 5.1 in the configuration of Figure 7 (i.e., the channel \(H\rho\) with preprocessor \(E'_1\) can achieve the same \(B_{\text{max}}\) with or without feedback). In other words,

\[
B^{FB}_{\text{max}}(HE_1, W, K) = B_{\text{max}}(H\rho E'_1, W, K) \quad (10)
\]

\[
= B_{\text{max}}(HE_1, W, K) \quad (11)
\]

for any given delay \(K\).

However, Eq. (10) is in contradiction with the assumption that \(E_1\) and \(E_2\) achieve better \(B_{\text{max}}\). Therefore, the only possible case is that \(\|E_1\| = \infty\) or \(E_1\) is unstable. Furthermore, in order to satisfy Eq. (4) we need \(E_2\) to be unstable as well. 

The above theorem is reminiscent of the results in [14] which show the equivalence between communication schemes based on receiver feedback and feedback stabilization through an analog communication channel using an unstable controller.

**Remark:** The above theorem has considerable implications on the realization of the preprocessor/feedback units. If we assume that \(E_1\) and \(E_2\) are integrated in one system as on the top of Figure 8 (i.e., the transmitter sends signal \(r\) to the receiver and \(E_2\) is used in the transmitter side), the resulting architecture is realizable (this is known as the “two-parameter compensator” in the literature [22]). However, if we disintegrate \(E_1\) and \(E_2\), and assume that \(E_2\) is utilized in the receiver to send back the signal \(\tilde{r} = E_2r\) as at the bottom of Figure 8, then this structure is not realizable due to the fact that \(E_2\) is unstable; in this case, \(\tilde{r}\) will diverge.

Note that the results reported here so far are independent from the receiver structure. In the next section we restrict our attention to a DFE receiver structure.
In this section, we consider a decision feedback equalizer (DFE) as our receiver. We also assume that the transmitter receives feedback from the receiver as depicted in Figure 9 where $Q$ and $D$ are respectively the feedforward and feedback filters of the DFE. Here, $\Theta$ is a thresholding operator that produces $-1$ or $1$ depending on which one has the closest distance to its input $\tilde{s}$; in this particular case, $(\Theta \tilde{s})(k) = \text{sgn} [\tilde{s}(k)]$. Notice that once again Condition (4) is imposed to constrain the transmission power.

Figure 9: Basic structure of DFE receiver with feedback.
Theorem 7.1 In Figure 9 perfect reconstruction is possible with delay $K$ for some $E := (E_1 \ E_2)$, $Q$ and $D = \Lambda F$ if and only if there exist $Q$, $F$, $E_1$ and $E_2$ in Figure 10 such that
\[
\| (s, \bar{n}) \rightarrow \Lambda^K s - \bar{s} \| < 1 \text{ and } \| (s, \bar{n}) \rightarrow u \| \leq 1.
\]
Moreover, if $E$, $Q$ and $D$ satisfy Condition (12), then they form a perfect reconstructing system in Figure 9.

Proof The “if” part follows from the fact that the signals $s$ and $\bar{n}$ are $\ell^\infty$ bounded (by 1) sequences and therefore Condition (12) guarantees that the soft error $(\Lambda^K s - \bar{s})(k)$ is always bounded away from 1 for all $k$. Hence, $\hat{s}(k) = \Theta \bar{s}(k) = s(k - K)$ for all time instants $k \geq K$.

For the “only if part” we need to show that if $E_1$ and $E_2$ satisfy the power Condition (4) and perfect reconstruction is possible for all $s \in \{-1, +1\}$ and $\bar{n} \leq b$ for some $Q$, $D$ in Figure 9, then there exist $Q$, $F$, $E_1$ and $E_2$ that satisfy Eq. (12).

It should be clear that for this structure to perfectly reconstruct $s(k)$ with delay $K$, it is necessary and sufficient that $\hat{s}(k) > 0$ if $s(k - K) > 0$ (and $\hat{s}(k) < 0$ if $s(k - K) < 0$) for each time step $k$ and all possible signal sequences $s$ and all noise sequences $n$.

Perfect signal reconstruction in Figure 9 means that if $s(k - K) = +1$ there are $E_1$, $E_2$, $Q$, $F$ and an (arbitrarily small) $\epsilon > 0$ such that
\[
\hat{s}(k) = (G^1_1 s + G^2_1 n)(k) > \epsilon > 0
\]
for all sequences $n$ with $\|n\| \leq b$, where $G^1 := (s \rightarrow \bar{s}) = \{g_0^1, g_1^1, \ldots\}$ and $G^2 := (n \rightarrow \bar{s}) = \{g_0^2, g_1^2, \ldots\}$ are the respective maps from $s$ to $\bar{s}$ and $n$ to $\bar{s}$ in Figure 10. The above condition is equivalent to
\[
\hat{s}(k) = \sum_{i=0}^{k} g_{k-i}^1 s(i) + \sum_{i=0}^{k} g_{k-i}^2 n(i) > \epsilon > 0
\]
whenever $s(k - K) = 1$ for all $n$ with $\|n\| \leq b$. Equivalently, we have that
\[
g_K^1 > \sum_{i=0, i \neq K}^{k} |g_i^1| + b \sum_{i=0}^{k} |g_i^2| + \epsilon \quad \forall \quad k = K + 1, K + 2, \ldots
\]

Notice that if $a := \|(G^1 \ G^2)\|$, then letting $\bar{G}^1 := a^{-1} G^1$ and $\bar{G}^2 := a^{-1} G^2$. By definition, we have that $\|(G^1 \ G^2)\| \leq 1$; hence
\[
\bar{g}_K^1 > \sum_{i=0, i \neq K}^{k} |\bar{g}_i^1| + b \sum_{i=0}^{k} |\bar{g}_i^2| + \epsilon' \quad \forall \quad k = K + 1, K + 2, \ldots
\]
where \( \epsilon' = \frac{\epsilon}{a} \). As \( 1 - |1 - \bar{g}_K| = \bar{g}_K^1 \) (notice that \( \bar{g}_K^1 \) is positive), we get that

\[
1 > |1 - \bar{g}_K| + \sum_{i=0, i \neq K}^k |\bar{g}_i^1| + b \sum_{i=0}^k |\bar{g}_i^2| + \epsilon' \quad \forall k = K + 1, K + 2, \ldots
\]

or \( \|(\Lambda^K - \bar{G}^1 G^2 b)\|_1 < 1 \).

**THIS MAKES NO SENSE TO ME! THIS MAKES NO SENSE TO ME! THIS MAKES NO SENSE TO ME! THIS MAKES NO SENSE TO ME! THIS MAKES NO SENSE TO ME!**

Note that if \( G^1 \) and \( G^2 \) is a result of \( E_1, E_2, Q \), then \( E_1, E_2, \tilde{Q} := a^{-1}Q \) and \( \tilde{F} := a^{-1}F \) will result \( \tilde{G}^1 \) and \( \tilde{G}^2 \). Hence, there exist \( E_1, E_2, \tilde{Q} := a^{-1}Q \) and \( \tilde{F} := a^{-1}F \) to satisfy Eq. (12).

Although the proof implicitly considered SISO systems, the same argument applies to MIMO systems by separately looking at each component.

The resulting optimization for selecting \( E_1, E_2, Q \) and \( D \) transforms to an \( \ell^1 \) performance problem in a closed loop system. We can view this in the standard context of controller design as shown in Figure 11. The generalized plant \( P \) is depicted in this figure along with the controller \( C \) that defines \( Q, F, E_1 \) and \( E_2 \); as it can be seen, this is a structured controller [7]. For perfect reconstruction, the closed loop maps \( \Phi_1 := (s, \tilde{n}) \rightarrow \Lambda^K s - \tilde{s} \) and \( \Phi_2 := (s, \tilde{n}) \rightarrow u \) should satisfy the \( \ell^1 \) constraints

\[
\|\Phi_1\| < 1 \text{ and } \|\Phi_2\| \leq 1.
\]

In general, the underlying \( \ell^1 \) optimization is not convex. Yet it can be viewed as a parameterized family of convex problems where the parameters are the first \( K + 1 \) coefficients of \( Q \). Indeed, \( Q \) can be parameterized as \( Q = Q_K + \Lambda^{K+1} \tilde{Q} \) where \( Q_K = \{q_0, \ldots, q_K, 0, 0, \ldots\} \) and \( \tilde{Q} \) is an arbitrary LTI system. Then, we can construct an equivalent loop by absorbing \( Q_K \) in the generalized plant.
as shown in Figure 12. The new generalized plant $P_{Q_K}$ is stabilized by

$$C_{Q_K} = \begin{pmatrix} E_1 & E_2 \\ -F & \bar{Q} \end{pmatrix},$$

which is a convex condition.

**Remark:** The case of arbitrary (but known) communication delay in the feedback loop can be handled similarly. Specifically, if there is an $M$-step delay in accurately passing to the transmitter the information about $r$ through the feedback path, then this is equivalent to requiring that $E_2 = E_3 \Lambda^M$. This structural condition on $C_{Q_K}$ leads to a convex problem for any fixed $Q_K$. This can be seen as follows. Define $P_{22}$ to be the map $(u, \sigma) \rightarrow (r, s)$ in the “open loop” plant $P_{Q_K}$; then,

$$P_{22} = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}.$$ 

All stabilizing controllers for $P_{22}$ and hence for $P_{Q_K}$ are given (see, for example, [22]) as $C_{Q_K} = Z(I + P_{22}Z)^{-1}$ where

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

is any stable (and causal) system. From this expression, it readily follows that $E_2 = E_3 \Lambda^M$ if and only if $Z_{12}$ is of the form $Z_{12} = Z_2 \Lambda^M$ where $Z_2$ is arbitrary and stable. In terms of this parametrization

$$\Phi_1 = T_{11} - T_{21}ZT_{31} \quad \text{and} \quad \Phi_2 = T_{12} - T_{22}ZT_{32},$$

where the $T_{ij}$'s are stable and depend on $H$, $Wb$ and $Q_K$. Thus, the relevant problem for perfect reconstruction is

$$\mu = \inf_{Z : Z_{12} = Z_2 \Lambda^M, \|T_{12} - T_{22}ZT_{32}\| \leq 1} \|T_{11} - T_{21}ZT_{31}\| < 1,$$

(14)
which is an $\ell^1$ optimization problem that corresponds to an infinite LP. These types of (structured) $\ell^1$ problems can be readily solved with available software [23] within any predefined accuracy.

![Diagram](image)

**Figure 12:** Controller design setup parameterized by $Q_K$.

We should also point out that the no feedback case is the special case when $E_2 = 0$. It is easy to see that this corresponds to having $Z_{12} = 0$ which leads again to a structured $\ell^1$ optimization that can be solved effectively [23] for any given parameter set \( \{q_0, \ldots, q_K\} \).

### 7.1 Investigation of Feedback Effectiveness

Recall that for a given noise bound $b$, the problem in Eq. (12) becomes

\[
J_b = \min_{E_1, E_2, Q, F} \| \Psi_1 \Psi_2 \| \\
\text{subject to } \|(I - E_2 H)^{-1} [E_1 E_2 Wb] \| \leq 1,
\]

where

\[
\Psi_1 = (s \rightarrow \Lambda^K s - \tilde{s}) \quad (s \rightarrow \Lambda^K s - \tilde{s}) = \Lambda^K - Q(I - HE_2)^{-1}HE_1 + \Lambda^{K+1}F
\]

\[
\Psi_2 = (\tilde{n} \rightarrow \Lambda^K s - \tilde{s}) = Q(I - HE_2)^{-1}Wb
\]

The ultimate problem is finding the *maximum tolerable noise bound*, i.e., the maximum bound $b$ such that $J_b < 1$.

Notice the difference between the concepts “perfect reconstructability margin” and “maximum tolerable noise bound”; the former depends only on the channel, the coloring filter and delay of reconstruction and can be found by indistinguishability arguments while the latter also depends
on the receiver structure (and is lower or equal to the former). In fact, in [3] we showed that in a SISO channel with no delay or one step delay, the maximum tolerable noise bound using a DFE is equal to the perfect reconstructability margin (with no preprocessing or feedback). Here, we study more general cases with preprocessing and/or feedback.

**Case I: Preprocessing without Feedback** to Corollary 3.1

**Proposition 7.1** *In the SISO case of preprocessing with no feedback* ($E_2 = 0$), *having a preprocessor does not help us tolerate larger noise bounds when using a DFE receiver.*

**Proof**  We show that if a DFE with preprocessing can tolerate noise bound $b^p$, then there exists another DFE without preprocessor that can tolerate the same noise bound. Since $E_2 = 0$, the problem in Eq. (15) becomes

$$
\min_{E_1, Q, F} \| \Lambda^K - QHE_1 + \Lambda^{K+1}F \cdot QWb \|
$$

subject to $\|E_1\| \leq 1$.  \hfill (18)

If $E^p_1, Q^p, F^p$ achieve maximum noise bound $b^p$ for the equivalent problem

$$
\mu = \min_{E_1, Q, F} \| \Lambda^K - QW^{-1}HE_1 + \Lambda^{K+1}F \cdot Qb \|
$$

subject to $\|E_1\| \leq 1$, \hfill (19)

we have

$$
\mu^p = \| \Lambda^K - Q^pW^{-1}HE^p_1 + \Lambda^{K+1}F^p \cdot Q^pb^p \|
\geq a \| \Lambda^K - Q^pW^{-1}HE^p_1 + \Lambda^{K+1}F^p \| + \|Q^pb^p\|
\geq b \| \Lambda^K - Q^pW^{-1}HE^p_1 + \Lambda^{K+1}F^p \| + \|Q^pb^p\| \cdot \|E^p_1\|
\geq c \| \Lambda^K - Q^pE^p_1W^{-1}H + \Lambda^{K+1}F^p \| + \|Q^pE^p_1b^p\|
$$

where (a) is due to the fact that the problem is SISO, (b) is a result of the power constraint $\|E_1\| \leq 1$, and (c) is a result of the sub-multiplicative inequality and the commutative property of SISO systems. Now, we can see that $Q_o = Q^pE^p_1$ and $F_o = F^p$ achieve $b^p$. In other words, there is a DFE without preprocessing that can perform as good as the mentioned DFE with preprocessing. □

**Remark:** Note that this argument cannot be generalized to the MIMO case because step (a) in the proof is not necessarily true for MIMO systems. To cope with the shortcomings of LTI
preprocessing for SISO channels, time-varying preprocessing scenarios were investigated in [16]. In particular, we discussed the design of periodic preprocessors as well as block preprocessing by lifting the input signal; in addition, we proposed a sub-optimal solution for DFE design through an \( \ell^1 \) iteration procedure.

**Case II: Feedback without Preprocessing**

We use similar approach in this case: if we focus on the case of no preprocessing (i.e., \( E_1 = I \)), the above problem in Eq. (15) can be stated as

\[
\begin{align*}
\min_{E_2,Q,F} & \quad \| \Lambda^K - Q(I - HE_2)^{-1}H + \Lambda^{K+1}F \\ & \quad Q(I - HE_2)^{-1}Wb \| \\
\text{subject to} & \quad \| (I - E_2H)^{-1}[1 \ E_2Wb] \| \leq 1.
\end{align*}
\]

If \( E_2^1, Q^1, F^1 \) achieve maximum noise bound \( b^1 \) using Eq. (21) we can see that the same \( b^1 \) can be achieved without any feedback from the receiver to the transmitter by using a DFE with feedforward filter \( Q_o = Q^1(I - HE_2^1)^{-1} \) and feedback filter \( F_o = F^1 \). The following proposition summarizes the results for this case.

**Proposition 7.2** If no preprocessing is used \( (E_1 = I) \), then feedback from the receiver to the transmitter will not increase the maximum tolerable noise bound when using a DFE receiver.

**Case III: Feedback with Preprocessing**

The following proposition summarizes the results with feedback and preprocessing when a DFE is utilized as the receiver structure.

**Proposition 7.3** With a DFE receiver, if preprocessing \( E_1 \) is norm bounded as \( \| E_1 \| \leq 1 \), then there is no pair of feedback \( E_2 \) and preprocessing \( E_1 \) that can improve the maximum tolerable noise bound over the maximum tolerable noise bound achieved by preprocessing \( E_1 \) alone \( (E_2 = 0) \).

**Proof** Assume \( E_1^f, E_2^f, Q^f, F^f \) achieve the maximum noise bound \( b^f \) in the problem of Eq. (15); if \( \| E_1^f \| \leq 1 \), then the DFE \( Q_o^f = Q^f(I - HE_2^f)^{-1} \), \( F_o = F^f \) along with same preprocessor \( E_1^f \) can achieve the same noise bound without feedback.

Note that the necessary condition for improving the perfect reconstructability margin in Theorem 6.1 (instability of \( E_1 \) and \( E_2 \)) may not necessarily hold for DFEs. Our numerical experiments did not result in any contradictory examples, i.e., we were not able to find stable \( E_1 \) and \( E_2 \) that improve the maximum tolerable noise bound achieved by DFE.
THIS MAKES NO SENSE. In the beginning of Section 7 we discussed the design problem of designing the DFE along with feedback/preprocessing and showed that the problem is non-convex.

For the numerical example in Section 5.2 (considering one step reconstruction delay $K = 1$). As it was discussed in the beginning of Section 7, we can obtain a DFE by first parameterizing the feedforward filter $Q$ (illustrated in Figure 12), using exhaustive search on $q_0$ and $q_1$ (for $Q_K = q_0 + q_1z^{-1}$), and solving the following convex problem:

$$J'_b = \min_{E_1, E_2, Q, F} \left\| \begin{pmatrix} s \\ \tilde{n} \end{pmatrix} \right\| \Lambda^K s - \tilde{s}$$

subject to $\left\| \begin{pmatrix} s \\ \tilde{n} \end{pmatrix} \right\| \rightarrow u \leq 1$;

then, we can use a bisection method to find the maximum $b$ that guarantees that $J'_b < 1$. For the example in Section 5.2, this procedure results in a DFE that consists of the following feedforward and feedback filters:

$$Q(z) = \frac{0.05 + 0.007674z^{-1} - 0.0039z^{-2} - 0.01z^{-3}}{1 - 4.041z^{-1} - 0.6347z^{-2} + 0.02832z^{-3}},$$

$$F(z) = \frac{0.4207 + 0.057z^{-1} - 0.0028z^{-2} + 0.0016z^{-3}}{1 - 4.041z^{-1} - 0.6347z^{-2} + 0.02832z^{-3}},$$

and along with the $E_1$ and $E_2$ of Eq. (8) achieves maximum tolerable noise bound $b = 15.625$. Although this bound is less than the perfect reconstructability margin obtained in Section 5.2 (which was 16.82), it is larger than the maximum tolerable noise bound that is achieved without feedback (which was 15).

Another point to be made is that the above filters share the unstable poles of $E_1$ and $E_2$ (at $z = 4.19$ in this example). This raises another issue regarding the implementation of this DFE receiver structure; this is addressed in the next section.

### 7.2 Implementation Issues

In the previous section we argued that the architecture of Figure 10, along with the design procedure there, may result to a solution in which $E_1, E_2, Q$ and $F$ are unstable with the same unstable poles. The results of Theorem 6.1 are in line with this observation and so is the case for our numerical example in the previous section.
To obtain an implementation, we first decompose the feedforward and feedback filter of the DFE to \( Q = UQ_1 \) and \( F = UF_1 \) where \( Q_1 \) and \( F_1 \) are stable and \( U \) is unstable. Using the fact that \( \hat{s} \) depends only on the sign of \( \hat{s} \), we replace \( U \) with a stable and possibly time-varying system that generates \( \hat{y} \) instead of \( \hat{s} \) but \( sgn(\hat{y}) = sgn(\hat{s}) \). This method is illustrated in Figure 13.

Now we discuss the details of this approach. Assume that the system \( U \) at the bottom of Figure 13 has the following state-space realization:

\[
\begin{align*}
    x(k + 1) &= Ax(k) + B\hat{r}(k) \\
    \hat{s}(k) &= Cx(k) + D\hat{r}(k).
\end{align*}
\]  

If we define \( \hat{y}(k) = f(k)\hat{s}(k) \) and \( z(k) = f(k)x(k) \) with \( f(k) > 0 \ \forall \ k \), then \( sgn(\hat{y}) = sgn(\hat{s}) \) and we can write

\[
\begin{align*}
    z(k + 1) &= Af^{-1}(k)f(k + 1)z(k) + Bf(k + 1)\hat{r}(k) \\
    \hat{y}(k) &= Cz(k) + Df(k)\hat{r}(k).
\end{align*}
\]  

If we let \( f(k) = \alpha^k \) where \( \alpha > 0 \) and \( \alpha \rho(A) < 1 \), then the above system will be equivalent to

\[
\begin{align*}
    z(k + 1) &= \alpha Az(k) + B\alpha^{k+1}\hat{r}(k) \\
    \hat{y}(k) &= Cz(k) + D\alpha^k\hat{r}(k),
\end{align*}
\]  

which is a stable but time-varying system. Note that due to the choice of \( \alpha \), this scheme will lead to \( \lim_{k \to \infty} \hat{y}(k) = 0 \), which is not desirable for our decision unit. Therefore, we change our choice for \( f(k) \) to

\[
f(k) = \begin{cases} 
    \alpha f(k - 1), & \hat{y}(k) > \epsilon \\
    \alpha^k, & \text{else},
\end{cases}
\]  

Figure 13: Top: Decomposition of the unstable DFE; Bottom: Equivalent stable but time-varying structure.
where \( f(0) = 1 \), parameter \( \epsilon \) is determined by the accuracy of the decision unit, and \( t_k \) is defined to be the last time step before \( k \) such that \( \tilde{y}(t_k) > \epsilon \).

8 Conclusions

In this paper, we consider transmitter-receiver systems that perfectly reconstruct discrete-valued data sequences, possibly with a given delay, under all possible realizations of channel noise that is limited in magnitude by a known bound. We allow norm-bounded preprocessing and feedback from the received signal to the transmitter, and discuss conditions under which the perfect reconstructability margin can be improved by the use of such feedback-preprocessor pairs. Specifically, we show that no stable feedback can satisfy that condition. The analysis for the case of more complicated alphabets, although not presented here, follows a similar path. The presented results are general in the sense that the analysis does not consider any particular receiver structure. For the special case of a DFE structure for the receiver (while imposing a linear transmitter structure together with the requirement that the power of the transmission is limited), we provide necessary and sufficient conditions for perfect reconstruction in terms of \( \ell^1 \) norms of appropriate maps. Finally, a procedure that results in parametric \( \ell^1 \) optimization is developed to optimize the design parameters for the transmitter-receiver pair in the cases where feedback from the receiver to the transmitter is available.

In future work, we plan to consider noisy feedback and investigate how that affects the results of this work.

References


