Detectability in Stochastic Discrete Event Systems

Christoforos Keroglou, Christoforos N. Hadjicostis

Department of Electrical and Computer Engineering, University of Cyprus

Abstract

A discrete event system possesses the property of detectability if it allows an observer to perfectly estimate the current state of the system after a finite number of observed symbols, i.e., detectability captures the ability of an observer to eventually perfectly estimate the system state. In this paper we analyze detectability in stochastic discrete event systems (SDES) that can be modeled as probabilistic finite automata. More specifically, we define the notion of A-detectability, which characterizes our ability to estimate the current state of a given SDES with increasing certainty as we observe more output symbols. The notion of A-detectability is differentiated from previous notions for detectability in SDES because it takes into account the probability of problematic observation sequences (that do not allow us to perfectly deduce the system state), whereas previous notions for detectability in SDES considered each observation sequence that can be generated by the underlying system even if this observation sequence can only be generated with very small probability. We discuss observer-based techniques that can be used to verify A-detectability, and provide associated necessary and sufficient conditions. We also prove that A-detectability is a PSPACE-hard problem.

Keywords: Stochastic discrete event systems, detectability, stochastic detectability.

1. Introduction and Motivation

Early instances of state estimation problems in discrete event systems appear in [1] and [2], both of which formulate the observability problem that requires perfect knowledge of the current state of the system. The state estimation problem is key in many applications involving complex systems. For example, opacity [3, 4] requires that a given set of states (with certain properties of interest) remain opaque (non-identifiable) based on the generated sequence of observations, regardless of the underlying activity in the system. Another related application is fault diagnosis [5, 6, 7] which requires discrimination (within a finite time interval following the occurrence of a fault) between the set of normal states (states that are possible under normal behaviour) and the set of faulty states (states that are possible under faulty behaviour), for every possible trace that can be executed in the system; disambiguation between these two sets of states requires state estimation techniques. A similar problem in stochastic discrete event systems (probabilistic finite automata) is the classification between two given models (hidden Markov models or probabilistic finite automata) [8, 9]. Classification is closely related to diagnosability if we can treat these models separately (i.e., no transition takes place from a state in one set (which can be thought of as the set of normal states) to a state in the other set (which can be thought of as the set of faulty states, in cases where the fault occurs at system initialization), or vice-versa).

An important task associated with state estimation is that of accurate characterization of the possible (compatible) current states following a (possibly long) ob-
The paper is organized as follows: in Section 2 we revisit notation on automata (nondeterministic finite automata and probabilistic finite automata), languages and Markov chains, and we recall detectability for discrete event systems. In this section we also discuss the verification of detectability in nondeterministic finite automata using either a deterministic observer construction or a detector construction. In Section 3 we introduce the notion of A-detectability and its associated necessary and sufficient conditions, and we establish that A-detectability is a PSPACE-hard problem. During this development of the material we also provide several examples. We conclude in Section 4 with some directions for future research.

2. Detectability for Discrete Event Systems

2.1. Notation on Languages and Automata

Let Σ be an alphabet (set of events) and denote by Σ* the set of all finite-length strings of elements of Σ (sequences of events), including the empty string ε (the length of a string s is denoted by |s| with |ε| = 0). A language L ⊆ Σ* is a subset of finite-length strings in Σ* [12] (i.e., sequences of events with the convention that the first event appears on the left). Given strings s, t ∈ Σ*, the string st denotes the concatenation of s and t, i.e., the sequence of events captured by s followed by the sequence of events captured by t. For a string s, s̄ denotes the prefix-closure of s, and is defined as s̄ = {t ∈ Σ* | ∃t′ ∈ Σ*|t′ = s|).

Definition 1. (Nondeterministic Finite Automaton (NFA)). A nondeterministic finite automaton is captured by G = (X, Σ, δ, X₀), where X = {x₁, x₂, ..., xₙ|} is the set of states, Σ is the set of events, δ : X × Σ → 2² is the nondeterministic state transition function, and X₀ ⊆ X is the set of possible initial states.

For a set Q ⊆ X and σ ∈ Σ, we define δ(Q, σ) = ∪q∈Qδ(q, σ); with this notation at hand, the function δ can be extended from the domain X × Σ to the domain X × Σ² in a routine recursive manner: δ(x, σs) := δ(δ(x, σ), s) for x ∈ X, s ∈ Σ* and σ ∈ Σ (note that δ(x, ε) := {x}). The behavior of G is captured by L(G) := {s ∈ Σ* | ∃x₀ ∈ X₀[δ(δ(x₀, s) ≠ ∅)]. We use L(G, x) to denote the set of all traces that originate from state x of G (so that L(G) = ∪x₀∈X₀ L(G, x₀)).

Definition 2. (Deterministic Finite Automaton (DFA)). A deterministic finite automaton is captured by D = (X, Σ, δ, x₀), where X = {x₁, x₂, ..., xₙ|} is the set of states, Σ is the set of events, δ : X × Σ → X is the (possibly partially defined) state transition function, and x₀ ∈ X is the initial state.
The function $\delta$ can be extended from the domain $X \times \Sigma$ to the domain $X \times \Sigma^*$ in the routine recursive manner:

$$\delta(x, os) = \begin{cases} \delta(x, o), & \text{if } \delta(x, o) \text{ is defined}, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

for $x \in X$, $s \in \Sigma^*$ and $\sigma \in \Sigma$ (note that in this case $\delta(x, e) := x$). The behavior of $D$ is captured by $L(D) := \{s \in \Sigma^* | \delta(x_0, s) \text{ is defined}\}$.

In general, only a subset $\Sigma_{obs}$ ($\Sigma_{obs} \subseteq \Sigma$) of the events can be observed, so that $\Sigma$ is partitioned into the set of observable events $\Sigma_{obs}$ and the set of unobservable events $\Sigma_{uobs} = \Sigma - \Sigma_{obs}$. The natural projection $P_{\Sigma_{obs}} : \Sigma^* \rightarrow \Sigma_{obs}$ can be used to map any trace executed in the system to the sequence of observations associated with it. This projection is defined recursively as $P_{\Sigma_{obs}}(os) = P_{\Sigma_{obs}}(\sigma)P_{\Sigma_{obs}}(s)$, $\sigma \in \Sigma$, $s \in \Sigma^*$, with

$$P_{\Sigma_{obs}}(\sigma) = \begin{cases} \sigma, & \text{if } \sigma \in \Sigma_{obs}, \\ e, & \text{if } \sigma \in \Sigma_{uobs} \cup \{e\}, \end{cases}$$

where $e$ represents the empty trace [12]. In the sequel, the subscript $\Sigma_{obs}$ in $P_{\Sigma_{obs}}$ will be dropped when it is clear from context. We denote an observation sequence of length $n$ as $\omega = \omega_1\omega_2...\omega_n$, where $\forall i, \omega_i \in \Sigma_{obs}$.

**Definition 3.** (Possible states following a sequence of observations ($R : 2^{\Sigma_X} \times \Sigma_{obs} \rightarrow 2^{\Sigma_X}$)). Suppose that a nondeterministic automaton $G = (X, \Sigma, \delta, X_0)$ starts from a set of possible states $X' \subseteq X$: the set of all possible states after observing $\omega \in \Sigma^n_t$ is $R(X', \omega) = \{x \in X | (\exists x' \in X')(\exists s \in \Sigma^*)(P(s) = \omega \wedge x \in \delta(x', s))\}$.

The projection of the language $L(G)$ of a nondeterministic automaton $G$ is defined as $P(L(G)) = \{P(s) | s \in L(G)\}$. Note that using Definition 3, the unobservable reach [12] can be expressed as $UR(X') = R(X', e)$.

**Definition 4.** (Probabilistic Finite Automaton (PFA)). A stochastic discrete event system (SDES) is modeled in this paper as a probabilistic finite automaton (PFA) $H = (X, \Sigma, p, \pi_0)$, where $X = \{x_1, x_2, \ldots, x_k\}$ is the set of states, $\Sigma$ is the set of events, $\pi_0$ is the initial-state probability distribution vector, and $p(x, \sigma | x')$ is the state transition probability defined for $x, x' \in X$, and $\sigma \in \Sigma$, as the probability that event $\sigma$ occurs and the system transitions to state $x'$ given that the system is in state $x'$.

We can assign a probability to each trace in $\Sigma^*$ with the interpretation that this value determines the probability of occurrence of this trace: if $Pr(x, s)$ denotes the probability that $s$ is executed in the system and the end state of the system is state $x$, then we can define for $\sigma \in \Sigma, s \in \Sigma^*$,

$$Pr(x_i, s) = \sum_{\sigma \in \Sigma} p(x_i, \sigma | x_i) Pr(x_i, s)$$

$$Pr(x_i, s) = \sum_{\sigma \in \Sigma} Pr(x_i, \sigma)$$

$$Pr(x_i, s) = \sum_{\sigma \in \Sigma} Pr(x_i, \sigma)$$

**Definition 5.** (Probability of an observation sequence ($\omega$)). Suppose we have a PFA $H = (X, \Sigma, p, \pi_0)$, and $\Sigma_{obs} \subseteq \Sigma$ is the set of observable events with respect to a natural projection map $P$. For any observation sequence $\omega = \omega_1\omega_2...\omega_n \in \Sigma_{obs}$, of length $n$, the probability of state $x_i \in X$ is $\pi_{\omega}(x_i) = \sum_{x \in \Sigma_{obs}, \omega(i) = x} p(x_i, \sigma | x_i) Pr(x_i, s)$, $where p_{\omega}(x_i) is the probability of occurrence of observation sequence $\omega$ leading to state $x_i \in X$. The probability of observation sequence $\omega$ is $\pi(\omega) = \sum_{\sigma \in \Sigma} \pi_{\omega}(x_i)$.

Note that if one of two strings $s$ and $t$ (with $Pr(s) = Pr(t) = \omega$) is a prefix of the other (say $s \in \tilde{t}$), then to obtain the probability $\pi_{\omega}$, we only include the probability of the prefix string.

An example of a PFA can be seen in Fig. 1. When $p(x, \sigma | x_i) = 0$, state $x_i$ is not reachable from state $x_i$ via event $\sigma$ (in the diagram representing the given PFA, we do not draw such transitions). Clearly, we have $\sum_{x_i \in X} \sum_{s \in \Sigma} p(x_i, \sigma | x_i) = 1, \forall x_i \in X$.

**Remark 1.** Given a PFA $H = (X, \Sigma, p, \pi_0)$ we can associate with it a unique NFA $G = (X, \Sigma, \delta, X_0)$ where the state transition function $\delta : X \times \Sigma \rightarrow 2^\Sigma$ is defined for $x_i \in X, \sigma \in \Sigma$ as $\delta(x_i, \sigma) = \{x_j | p(x_j, \sigma | x_i) > 0\}$, and the set of possible initial states is defined as $X_0 = \{x_i | \pi_0(x_i) > 0\}$. In this way, the behavior of the PFA $H$ is mapped to the behavior of the associated NFA $G$, i.e., $L(S) = L(G)$ (where $L(S) = \{s \in \Sigma^* | Pr(s) > 0\}$).

**Definition 6.** (Markov chain $M$). Given a PFA $H = (X, \Sigma, p, \pi_0)$ we can associate with it a Markov chain $M = (X, T, \pi_0)$, where $X$ is the set of states, $T$ is the state transition matrix defined so that its $(k, j)$th entry captures the probability of a transition from state $x_k$ to state $x_j$ (also denoted by $p_{kj}(k, j) = \sum_{\sigma \in \Sigma} p(x_k, \sigma | x_j)$), and $\pi_0$ is the initial state probability distribution vector.
Given a Markov chain \( MC = (X, T, \pi_0) \), its state at time step \( n \) is denoted by \( x[n] \in X \). Specifically, \( \Pr(x[0] = x_i) = \pi_0(i) \) (where \( \pi_0(i) \) is the \( i \)th entry of vector \( \pi_0 \)) and the sequence of states \( x[0] = x_0, x[1] = x_1, \ldots, x[n] = x_n \) occurs with probability \( \Pr(x[0] = x_0, x[1] = x_1, \ldots, x[n] = x_n) = \pi_0(i_0)t_0(i_1, i_0) \cdots t_{n-1}(i_n, i_{n-1}) \) (where \( t(i, j) = \rho_M(i, j) \) is the \((i, j)\)th entry of the transition matrix \( T \)). The state probability distribution vector at step \( n \) is denoted by \( \pi_n \) (or \( \pi[n] \)) and it is a vector whose \( j \)th entry is the probability that the Markov chain is in state \( x_j \) after \( n \) steps (i.e., \( \Pr(x[n] = x_j) = \pi_n(j) \)). The vector \( \pi_n \) can be calculated recursively as \( \pi_n = T\pi_{n-1} \), with \( \pi_0 \) being the initial state probability distribution vector.

**Definition 7.** (Recurrent and transient states of a finite state Markov chain) [13]. Given a finite-state Markov chain \( MC = (X, T, \pi_0) \), a state \( x_i \) is said to be recurrent iff \( \sum_{n=0}^{\infty} \pi_i^n = \infty \), where \( \pi_i^n = \Pr(x[n] = i \mid x[0] = i) \) (note that \( \pi_i^n \) is the \( i \)th entry of vector \( \pi_n = T^n\pi_0 \), where \( \pi_0 \) is a column vector of length \( |X| \) and with a single one at its \( i \)th entry). State \( x_i \) is transient (if it is not recurrent).

**Example 1.** The following example is used to clarify the notion. Consider the PFA \( H \) depicted on the left in Fig. 1 with \( X = \{x_1, x_2, x_3\} \), \( \Sigma = \{(a, \beta), \delta\} \) as defined by the transitions in the figure (along with their probabilities), and \( \pi_0 = \left[ \begin{array}{c} 1 \\ 0.5 \\ 0.5 \end{array} \right] \) (i.e., each state is equally likely at the initialization of the system). Consider also the underlying Markov chain \( M \) of PFA \( H \), at the right of Fig. 1. The unique NFA \( G = (X, \Sigma, \delta, X_0) \) associated with PFA \( H \) has \( \delta \) as defined by the transitions in Fig. 2, and \( X_0 = \{x_1, x_2, x_3\} \).

3. A-detectability for stochastic discrete event systems

In this section we introduce the notion of A-detectability; we also develop a methodology to verify it using observer based techniques, and prove that A-detectability is a PSPACE-hard problem.

**Definition 9.** (A-Detectability). A stochastic discrete event system captured by PFA \( H = (X, \Sigma, p, \pi_0) \) is A-detectable from initial probability distribution \( \pi_0 \) with respect to a set of observable events \( \Sigma_{obst} \subseteq \Sigma \) if

\[
(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall s \in \Sigma^* : ||s|| = n \geq N \Rightarrow |R(X_0, P(s))| > 1)) < \varepsilon,
\]

where \( R(X_0, P(s)) \) is taken with respect to the NFA \( G \) associated with PFA \( H \).

**Remark 2.** It is worth discussing a bit differences and similarities between A-detectability and A-diagnosability [14], which is a similar notion in fault diagnosis. The main idea in both notions is that the most probable observation sequences allow us to resolve a specific property of the system with always increasing certainty. The difference between the two notions is that A-detectability resolves the exact state of the system (the ambiguity occurs when the state estimate involves at least two different states of the system), whereas in...
A-diagnosability the ambiguity for an observation sequence occurs when the state estimate involves at least one state from two different sets of states (the normal set of states, which represents the normal behaviour of the system, and the faulty set of states, which indicate that the fault has occurred). Another difference is the monotonicity property which is present in fault diagnosis but not in detectability. More specifically, when the state estimate involves only states in the set of the faulty states, then for any new observation the state estimate remains in the set of faulty states, and the fault diagnosis problem is resolved. This is not true in general for the detectability problem, because even if the state estimate is a single state at a certain point, it is possible that a new observation may drive the estimator to a state estimate that involves multiple states; thus, the problem of exact detection of a state is not resolved.

Remark 3. A comparison between Strong Detectability and A-detectability shows that if the NFA associated with a given PFA is strongly detectable then the PFA is A-detectable. The proof is outlined below. Suppose the nondeterministic automaton $G = (X, \Sigma, \delta, X_0)$, associated with a PFA $H = (X, \Sigma, p, \pi_0)$, is strongly detectable with respect to natural projection map $P$, the observer (or current-state estimator) is a deterministic finite automaton (DFA) with unobservable self-loops $(\hat{G}_{\text{obs}})$. Given an NFA $G = (X, \Sigma, \delta, X_0)$ with set of observable events $\Sigma_{\text{obs}} \subseteq \Sigma$ under the natural projection map $P$, the observer (or current-state estimator) is a deterministic finite automaton (DFA) $G_{\text{obs}} = (Q_{\text{obs}}, \Sigma_{\text{obs}}, \delta_{\text{obs}}, Q_{0,\text{obs}})$, which captures the state estimate (following a sequence of observations $\omega \in \Sigma_{\text{obs}}$) as follows:

(1) Each state of $G_{\text{obs}}$ is associated with a unique subset of states of the original NFA $G$ (this means that $Q_{\text{obs}} \subseteq 2^X$ has at most $2^N$ states).

(2) The initial state $Q_{0,\text{obs}}$ of $G_{\text{obs}}$ is the unobservable reach of $X_0$ ($Q_{0,\text{obs}} = UR(X_0) = R(X_0, \epsilon)$).

(3) At any state $Q \in Q_{\text{obs}}$ of the current-state estimator, the next state for any $\sigma \in \Sigma_{\text{obs}}$ is captured by $\delta_{\text{obs}}(Q, \sigma) = R(Q, \sigma)$ (Definition 3).

Note that the state associated with the empty subset of $X$ is typically excluded from the construction. In such case, the language of the observer is the same as the projection of the language of $G$ (i.e., we have $L(G_{\text{obs}}) = P(L(G))$).

Example 2. Consider NFA $G$ in Fig. 2 (associated with PFA $H$ in Fig. 1), and assume that $\Sigma_{\text{obs}} = \{\alpha, \beta\}$ and $\Sigma_{\text{uno}} = \emptyset$. We construct its observer (shown on Fig. 2).

Next we describe a useful extension of observer $G_{\text{obs}}$ based on the NFA $G$ that is associated with the given PFA $H$.

Definition 10. (Observer or Current-state estimator) [12]. Given an NFA $G = (X, \Sigma, \delta, X_0)$ with set of observable events $\Sigma_{\text{obs}} \subseteq \Sigma$ under the natural projection map $P$, the observer (or current-state estimator) is a deterministic finite automaton (DFA) $G_{\text{obs}} = (Q_{\text{obs}}, \Sigma_{\text{obs}}, \delta_{\text{obs}}, Q_{0,\text{obs}})$ constructed as in Definition 10. Adding a self-loop to each state of DFA $G_{\text{obs}}$ for each label in the set $\Sigma_{\text{uno}} = \Sigma - \Sigma_{\text{obs}}$, we create the DFA $G_{\text{obs}} = (Q_{\text{obs}}, \Sigma, \delta_{\text{obs}}, Q_{0,\text{obs}})$. More specifically $\delta_{\text{obs}}$ extends $\delta_{\text{obs}}$, as follows: for $Q \in Q_{\text{obs}}$ and $\sigma \in \Sigma$

\[
\delta_{\text{obs}}(Q, \sigma) = \begin{cases} 
\delta_{\text{obs}}(Q, \sigma), & \text{if } \sigma \in \Sigma_{\text{obs}}, \\
Q, & \text{if } \sigma \in \Sigma_{\text{uno}}.
\end{cases}
\]

Definition 11. (Stochastic Observer $H_{\text{obs}}$). Given a PFA $H = (X, \Sigma, p, \pi_0)$ and a natural projection map $P$ with respect to the set of observable events $\Sigma_{\text{obs}} \subseteq \Sigma$, $H_{\text{obs}}$ is constructed as follows:

(1) We construct the (deterministic) observer $G_{\text{obs}} = (Q_{\text{obs}}, \Sigma_{\text{obs}}, \delta_{\text{obs}}, Q_{0,\text{obs}})$, and then $\hat{G}_{\text{obs}} = (Q_{\text{obs}}, \Sigma_{\text{obs}}, \delta_{\text{obs}}, Q_{0,\text{obs}})$.
\(\delta_{\text{obs}}, Q_{0,\text{obs}}\) with respect to the NFA \(G = (X, \Sigma, \delta, X_0)\) associated with \(H\).

(2) We construct the PFA \(H_{\text{obs}} = H \times \hat{G}_{\text{obs}} := (X \times Q_{\text{obs}}, \Sigma, p_{\text{obs}}, \pi_{\text{obs}})\), where \(X \times Q_{\text{obs}}\) is the set of states, \(p_{\text{obs}}(x'_j, \sigma|x_i)\) is the state transition probability defined for \(x'_j = (x_j, Q_i) \in X \times Q_{\text{obs}}\) and \(x'_j \in X \times Q_{\text{obs}}\) (i.e., \(x_j \in X, Q_i \in Q_{\text{obs}}, x'_j \in X \text{ and } Q_i \in Q_{\text{obs}}\) and \(\sigma \in \Sigma\) as \(p_{\text{obs}}(x'_j, \sigma|x_i) = p(x_j, \sigma|x_i)\) if \(Q_i = \delta_{\text{obs}}(Q_i, \sigma)\), and \(p_{\text{obs}}(x'_j, \sigma|x_i) = 0\), otherwise; \(\pi_{\text{obs}}\) is the initial-state probability distribution vector given by a column vector with \(\pi_{\text{obs}}(x'_i) = \pi(x_i)\) if \(Q_i = Q_{0,\text{obs}}\) and zero otherwise.

**Definition 13.** (Markov chain MC of stochastic observer \(H_{\text{obs}}\)). Given a PFA \(H = (X, \Sigma, p, \pi_0)\), its associated NFA \(G = (X, \Sigma, \delta, X_0)\), and its deterministic observer \(G_{\text{obs}} = (Q_{\text{obs}}, \Sigma, \delta_{\text{obs}}, \pi_{\text{obs}})\), we construct the stochastic observer \(H_{\text{obs}} = H \times \hat{G}_{\text{obs}} := (X \times Q_{\text{obs}}, \Sigma, p_{\text{obs}}, \pi_{\text{obs}})\). The Markov chain \(MC = (X \times Q_{\text{obs}}, T_{\text{obs}}, \pi_{\text{obs}})\) associated with the PFA \(H_{\text{obs}}\) is the Markov Chain with state transition probabilities \(p_{\text{obs}}(x'_j, x'_i) = \sum_{\sigma \in \Sigma} p_{\text{obs}}(x'_j, \sigma|x'_i)\) for \(x'_j, x'_i \in X \times Q_{\text{obs}}\) (indexing the states in the order \((x_1, x_2, \ldots, x_i, \ldots, x_{\text{obs}})\)). The entries of the state transition matrix \(T_{\text{obs}}\) are given by \(T_{\text{obs}}(i, j) = p_{\text{obs}}(x'_j, x'_i)\).

**Example 3.** Given PFA \(H\) in Fig. 1 and \(G_{\text{obs}}\) in Fig. 2, the corresponding \(H_{\text{obs}}\), with \(Q_1 = \{x_1, x_2, x_3\}, Q_2 = \{x_2, x_3\}, Q_3 = \{x_3\}, Q_4 = \{x_2\}\), is as shown in Fig. 3. Ordering the states from left-right, and top-bottom as \((x_1', x_2', x_3', x_2', x_3', x_3', x_2', x_3')\), the state transition probability matrix of the underlying Markov chain is

\[
A_{\text{obs}} = \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0
\end{bmatrix}
\]

We now recall a useful property for a finite state Markov chain (see, for example, [13, 15]).

**Lemma 1.** Let \(X\) be the finite state space of Markov chain \(MC = (X, \Sigma, \pi_0)\) and \(X = X_R \cup X_T\), where \(X_R\) and \(X_T\) denote the non-intersecting sets of recurrent and transient states. We have that

\[
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) \quad \pi_n(x) \leq \sum_{x \in X_R} \pi_n(x) < \epsilon,
\]

which clearly implies that for any \(x_j \in X_T\), \(\pi_n(x_j) < \epsilon\).

**Remark 4.** Lemma 1 implies that as the number of transitions increases, the probability of the Markov chain being in a transient state approaches zero. If a state \(x_i \in X_R\) (equivalent to \(x_i \notin X_T\)) then it is easily proved (Lemma 1) that

\[
(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \geq N) \quad \pi_n(x_i) \geq \epsilon,
\]

as long as the recurrent states are reachable (with a nonzero probability) from the set of possible initial states (this can be easily ensured by trimming all states that are not reachable from possible initial states of the given Markov chain).

**Theorem 1.** (A-detectability using stochastic observer \(H_{\text{obs}}\). Necessary and sufficient conditions). Given a PFA \(H = (X, \Sigma, p, \pi_0)\), its associated NFA \(G = (X, \Sigma, \delta, X_0)\), we construct its observer \(G_{\text{obs}} = (Q_{\text{obs}}, \Sigma, \delta_{\text{obs}}, \pi_{\text{obs}})\), its stochastic observer \(H_{\text{obs}} = H \times \hat{G}_{\text{obs}} = (X \times Q_{\text{obs}}, \Sigma, p_{\text{obs}}, \pi_{\text{obs}})\), and its associated NFA \(G_{\text{obs}} = (X \times Q_{\text{obs}}, \Sigma, \delta_{\text{obs}}, \pi_{\text{obs}}, \pi_{\text{ols}})\), and Markov chain \(MC = (X \times Q_{\text{obs}}, T_{\text{obs}}, \pi_{\text{obs}})\) as in Definition 12.

Then, PFA \(H\) is \(A\)-detectable if the Markov chain \(MC\) has the following property:

\[
(\forall x_j' \equiv (x_j, Q_i) \in X_R \subseteq X \times Q_{\text{obs}})(\forall Q_i | Q_i = 1),
\]

where \(X_R\) is the set of recurrent states of Markov chain \(MC\) as defined in Lemma 1.

Theorem 1 implies that for a PFA to be \(A\)-detectable, we need all recurrent states \(x_j' \in X \times Q_{\text{obs}}\) of its underlying Markov chain to be associated with state estimates that involve a single state (i.e., have \(|Q_i| = 1\).

Proof. (⇒): Suppose there exists at least one recurrent state \(x_j' = (x_j, Q_i), Q_i \in Q_{\text{obs}}, \) with \(|Q_i| > 1\).

Clearly, \(x_j' \in X_{\text{R}}\) means that \((\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n \geq N)\) such that \(p(x_j, \Sigma^* : ||s|| = n \geq 1)\).
In this section we prove that A-detectability is PSPACE-hard by introducing a polynomial-time reduction of the language universality problem of an NFA to an instance of the A-detectability problem. Suppose that we are given an instance of the universality problem for \( G_T = (X_T, \Sigma, \delta_T, X_T) \). We construct the following instance of the A-detectability problem (refer to Fig. 14). PFA \( H = (X, \Sigma, p, \pi_0) \) has \( X = X_T \cup \{x_R\} \) where the set of states \( X_T = \{x_1, x_2, ..., x_T\} \) and \( x_R \) is a new state (not in \( X_T \)). The set \( \Sigma = \Sigma_o \cup \{\delta_{uo}\} \) is the set of events, where \( \Sigma_o \) is the set of observable events (events of \( G_T \)) with \( |\Sigma_o| \geq 2 \) and \( \delta_{uo} \) is a new event (not in \( \Sigma_o \)) that is unobservable. We assign probabilities as follows:

a) \( \forall x_i \in X_T, \forall x_T \in X \) and \( \sigma \in \Sigma \), if \( x_T \in \delta(x_i, \sigma) \), then
\[
\Pr(x_T, \sigma | x_i) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \delta(x_i, \sigma),
\]
otherwise
\[
\Pr(x_T, \sigma | x_i) = 0;
\]

b) \( \forall \sigma \in \Sigma_o, \Pr(x_R, \sigma | x_R) = \frac{1}{|\Sigma|} \).

Remark 6. Note that i) the set of states \( X_T \) is the set of transient states of PFA \( H \) and state \( x_R \) is the only recurrent state; ii) there exists a transition (via the unobservable event) from every state \( x_i \in X_T \) to state \( x_R \).

The following theorem shows that every instance of the language universality problem of an NFA \((G_T = (X_T, \Sigma, \delta_T, X_T))\) with all states initial, can be reduced to an instance of A-detectability problem (as it was described in previous paragraphs). Thus, the A-detectability problem is PSPACE-hard.

**Theorem 2.** A-detectability is PSPACE-hard.
We argue that language universality of NFA $G_T$ is equivalent to PFA $H$ (as in Fig. 4 and as described above) not being A-detectable.

$(\rightarrow)$ If $L(G_T) = \Sigma_n^\omega \Rightarrow (\forall s : |R(X, P(s)) > 1) \Rightarrow (\forall N \in \mathbb{N})(|s| = n \geq N) Pr(s : |R(X, P(s)) > 1) = 1$. This means that the system $H$ is not A-detectable.

$(\leftarrow)$ If $L(G_T) \neq \Sigma_n^\omega \Rightarrow (\exists K \in \mathbb{N})(\exists s : P(s) = \omega \in \Sigma_n^K : \omega \notin L(G_T))$. Notice that $\forall \omega \in \Sigma_n^\omega : \omega \in \omega \notin L(G_T)$, because $\delta(X_T, \omega_1 \omega) = \delta(\delta(X_T, \omega_1), \omega) \subseteq \delta(X_T, \omega) = \emptyset$.

Let $N' = NK$, where $N \in \mathbb{N}$. Then, $Pr(s : |P(s)| = n' \geq N' \wedge |R(X, P(s)) | > 1) \leq Pr(s : P(s) = \omega = \omega^{(1)} \omega^{(2)} ... \omega^{(N)} (with ||\omega|| = K and ||\omega|| = N') \wedge \omega_j = \omega_j \forall j = 1, 2, ..., N)) \leq (1 - \frac{\epsilon}{|\omega||\Sigma||})^N \Rightarrow (\forall \epsilon > 0)(\exists N' = NK \in \mathbb{N}) Pr(s : |s| = n' \geq N' \wedge |R(X, P(s)) | > 1) < \epsilon$ (namely, $N' = NK$, with $N \geq \lceil \log(\frac{1}{\epsilon}) \rceil$).

This means that $H$ is A-detectable.

This establishes the reduction and we conclude that the A-detectability problem is PSPACE-hard.

4. Conclusions

In this paper we studied detectability in discrete event systems modeled by PFA. We defined and analyzed a notion of stochastic detectability, namely A-detectability, which was inspired by analogous notions in stochastic diagnosability [14]. We showed that A-detectability is a PSPACE-hard problem and applied observer-based techniques to verify it. Interesting future research directions include the introduction and verification of stochastic notions for detectability that will ensure that the probability of error in state estimation approaches zero for very large observation sequences, as well as the computation of bounds on the probability of error in classification problems.

References