Distributed Balancing of Commodity Networks
Under Flow Interval Constraints

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Abstract—We consider networks the nodes of which are interconnected via directed edges, each able to admit a flow (or weight) within a certain interval, with nonnegative end points that correspond to lower and upper flow limits. The paper proposes and analyzes a distributed algorithm for obtaining admissible and balanced flows, i.e., flows that are within the given intervals at each edge and are balanced (the total in-flow equals the total out-flow) at each node. The algorithm can also be viewed as a distributed method for obtaining a set of weights that balance a weighted digraph for the case when there are lower and upper limit constraints on the edge weights. The proposed iterative algorithm assumes that communication among pairs of nodes that are interconnected is bidirectional (i.e., the communication topology is captured by the undirected graph that corresponds to the network digraph), and allows the nodes to asymptotically (with geometric rate) reach a set of balanced feasible flows, as long as the (necessary and sufficient) circulation conditions on the given digraph, with the given flow/weight interval constraints on each edge, are satisfied. We also provide a methodology that can be used by the nodes to determine, in a distributed manner, when the circulation conditions are not satisfied (thus, making the problem infeasible). Finally, we provide several examples and simulation studies to highlight the role of the various parameters involved in the proposed distributed algorithm.

I. INTRODUCTION

We consider a system comprised of multiple nodes that are interconnected via some directed links through which a certain commodity can flow. We assume that the flow on each link is constrained to lie within an interval the end points of which are nonnegative, corresponding to link lower and upper capacity limits. The objective is to find a feasible flow assignment i.e., find flows on all the links that are within the corresponding capacity limits and balance each node, i.e., the sum of in-flows is equal to the sum of out-flows. In this paper, we propose an iterative algorithm that allows the nodes to distributively compute a solution to this feasibility problem.

The problem of interest in this paper is a particular case of the standard network flow problem (see, e.g., [2]), where there is a cost associated to the flow on each link, and the objective is to minimize the total cost subject to the same constraints in the flow assignment problem described above.

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Parts of the results in this paper appeared in [1]. This extended version provides complete proofs of the various theorems and propositions, a distributed algorithm for determining feasibility, as well as simulations/comparisons against previous work; all of these were not included in [1].

In such settings, it is common to assume that the individual costs are described by convex functions on the flow, which makes the optimization problem convex. Then, its solution can be obtained via the Lagrange dual, the formulation of which is well suited for algorithms that can be executed, in a distributed fashion, over a network that conforms to the same topology as that of the multi-node system (see, e.g., [3]); however, recovering the optimal primal solution from the dual one might not be straightforward [4].

By contrast, the distributed algorithm proposed in this paper does not exploit duality notions, and instead acts directly on the primal variables, i.e., the flows. In this regard, it can be shown that the algorithm is a gradient descent algorithm for a quadratic optimization program. In this program, the flows are constrained to lie within the corresponding link lower and upper capacity limits; and the cost function is the two-norm of the projection of the balance vector (the entries of which are the differences of the nodes in- and out-flows) onto the positive orthant. Here, it is important to note that finding a feasible flow assignment is equivalent to finding a zero-cost solution to this quadratic program. Also, if the solution of the quadratic program has a nonzero cost, then there is no solution to the flow assignment problem. [The quadratic program is always feasible as long as the set defined by lower and upper capacity limits is non-empty.] In terms of establishing convergence of our proposed algorithm, one could attempt to utilize off-the-shelf convergence results for optimization problems (see, e.g., [5]). However, with these results one can only establish optimality of all limits points of the sequence generated by our algorithm, but one cannot establish convergence; to address this issue, we utilize an alternative proof technique that relies on showing that (i) the set of nodes with a positive balance enlarges monotonically (in a way that includes all nodes that previously had positive balance) as the algorithm progresses, reaching a set that includes all but one vertices, as long as a feasible flow assignment exists; and (ii) the sum of positive balances decreases monotonically. Then, since the sum of the balances must always be zero, this means that all balances converge to zero asymptotically as long as a feasible flow assignment exists.

The problem we deal with in this paper can also be viewed as the problem of weight balancing a given digraph. A weighted digraph is a digraph in which each edge is associated with a positive real or integer value, called the edge weight. A weighted digraph is weight-balanced or balanced if, for each of its nodes, the sum of the weights of the edges outgoing from the node is equal to the sum of the weights of the edges incoming to the node.
Weight-balanced digraphs find numerous applications in control, optimization, economics and statistics. Examples of applications where balance plays a key role include modeling of flocking behavior [6], network adaptation and synchronization strategies [7], distributed adaptive strategies to tune the coupling weights of a network based on local information of node dynamics [8], and prediction of distribution matrices for telephony traffic [9], or financial transactions [10]. It is also worth pointing out that weight balance is closely related to weights that form a doubly stochastic matrix [11], which find applications in multicomponent systems (such as sensor networks) where one is interested in distributively averaging measurements at each component. In particular, the distributed average consensus problem has received significant attention from the computer science community [12], and the control community [13] due to its applicability to diverse areas, including multi-agent systems, distributed estimation and tracking [14], distributed optimization [15], and coordination of distributed energy resources in electrical energy systems [16], [17].

Recently, quite a few works have appeared dealing with the problem of designing distributed algorithms for balancing a strongly connected digraph, for both real- and integer-weight balancing, for the case when there are no constraints on the edge weights in terms of the nonnegative values they admit [11]–[21]. Thus, one important difference in this paper is the presence of interval constraints on the link weights. We should point out that, unlike [11]–[21], in which it is assumed that there exists a communication topology that matches the flow (physical) topology, the algorithm developed in this paper requires a bi-directional communication topology. However, there are many applications where the physical topology is directed but the communication topology is bi-directional (e.g., traffic flow in an one way street is directional, but communication between traffic lights at the end points of the street will, in fact, be bi-directional). More generally, in many applications, the communication topology does not necessarily match the physical one; in our future work, we plan to enhance the algorithm proposed here to allow for different communication topologies (including the one that matches the physical topology).

The remainder of this paper is organized as follows. Section II provides some background on graph-theoretic notions used throughout the paper, the formulation of the constrained flow assignment problem in commodity networks, and a well-known result on necessary and sufficient conditions for the existence of solutions to the constrained flow assignment problem. In Section III, we propose a distributed algorithm to solve the aforementioned constrained flow balancing problem and state the main convergence results. Section IV establishes some ancillary results that are needed for the proof of convergence of the proposed algorithm, which is provided in Section V. In Section VI, we provide an enhancement to the proposed algorithm that allows nodes to distributively detect whether or not there is a feasible solution to the constrained flow balancing problem. Simulation results showcasing the algorithms are presented in Section VII. Concluding remarks are presented in Section VIII.

II. MATHEMATICAL BACKGROUND AND NOTATION

In this section, we first provide some graph-theoretic notions used throughout the paper. Then, we formulate the constrained flow feasibility problem, i.e., the problem of finding a set of flows in a commodity network that satisfy some equality constraints associated to each node, and some interval constraints associated with edge capacity limits. We finish the section by introducing a well-known result in the network flow literature pertaining necessary and sufficient conditions for the existence of solutions to the constrained flow feasibility problem.

A. Graph-Theoretic Notions

A digraph (directed graph) of order \( n \) \((n \geq 2)\), is defined as \( G_d = (V,E) \), where \( V = \{ v_1, v_2, \ldots, v_n \} \) is the set of nodes and \( E \subseteq V \times V - \{(v_j, v_j) \mid v_j \in V\} \) is the set of edges. A directed edge from node \( v_i \) to node \( v_j \) is denoted by \((v_i, v_j) \in E\), and indicates a nonnegative flow from node \( v_i \) to node \( v_j \). We will refer to the digraph \( G_d \) as the flow topology.

Note that the definition of \( G_d \) excludes self-edges.

We assume that a pair of nodes \( v_j \) and \( v_l \) is connected by an edge in the digraph \( G_d \) (i.e., \((v_j, v_l) \in E\), and/or \((v_l, v_j) \in E\)) can exchange information among themselves. In other words, the communication topology is captured by the undirected graph \( G_u = (V, \mathcal{E}_u) \) that corresponds to a given digraph \( G_d = (V, E) \), where \( \mathcal{E}_u = \cup \{(v_j, v_l) \mid (v_l, v_j) \in E\} \). [Recall that a graph is undirected if and only if \((v_j, v_l) \in E \) implies \((v_l, v_j) \in E\).]

A digraph is called strongly connected if for each pair of vertices \( v_j, v_l \in V \), \( v_j \neq v_l \), there exists a directed path from \( v_l \) to \( v_j \) i.e., we can find a sequence of vertices \( v_l = v_{n_0}, v_{n_1}, \ldots, v_{n_t} = v_j \) such that \((v_{n_{i+1}}, v_{n_i}) \in E\) for \( \tau = 0, 1, \ldots, t-1 \). All nodes from which node \( v_j \) can receive flows are said to be in-neighbors of node \( v_j \) and belong to the set \( N_j^- = \{ v_l \in V \mid (v_l, v_j) \in E\} \). The cardinality of \( N_j^- \) is called the in-degree of \( j \) and is denoted by \( D_j^- \). The nodes that receive flows from node \( v_j \) comprise its out-neighbors and are denoted by \( N_j^+ = \{ v_l \in V \mid (v_j, v_l) \in E\} \). The cardinality of \( N_j^+ \) is called the out-degree of \( v_j \) and is denoted by \( D_j^+ \). We also let \( D_j = D_j^+ + D_j^- \) denote the total degree\(^1\) of node \( v_j \) and \( N_j = N_j^+ \cup N_j^- \) denote the neighbors of node \( v_j \).

B. Network Flows and Problem Formulation

A flow commodity network can be described by a digraph \( G_d = (V, \mathcal{E}) \), with nonnegative flows (sometimes, also viewed as weights) \( f_{ji} \in \mathbb{R} \) associated with each edge \((v_j, v_l) \in \mathcal{E}\).

In this paper, these flows will be restricted to lie in a real interval \([l_{ji}, u_{ji}]\), \( 0 < l_{ji} \leq f_{ji} \leq u_{ji} \). We will also use matrix notation to denote (respectively) the flow, lower limit, and upper limit matrices by the \( n \times n \) matrices \( F = [f_{ji}] \), \( L = [l_{ji}] \), and \( U = [u_{ji}] \), where \( F(j,i) = f_{ji}, L(j,i) = l_{ji}, \) and \( U(j,i) = u_{ji} \) (and \( F(j,i) = L(j,i) = U(j,i) = 0 \) when \((v_j, v_l) \notin \mathcal{E}\)).

\(^1\)Note that the total degree \( D_j \) counts twice each node that is both an in-neighbor and an out-neighbor of node \( v_j \). Thus, in a graph of \( n \) nodes, \( D_j \) is bounded by \( 2(n - 1) \).
**Definition 1:** Given a digraph $G_d(V, \mathcal{E})$ of order $n$, along with a flow assignment $F = \{f_{ji}\}$, the total in-flow of node $v_j$ is denoted by $f_i^-$, and is defined as $f_i^- = \sum_{v_i \in N^-_i} f_{ji}$, whereas the total out-flow of node $v_j$ is denoted by $f_i^+$, and is defined as $f_i^+ = \sum_{v_i \in N^+_i} f_{ji}$.

**Definition 2:** Given a digraph $G_d(V, \mathcal{E})$ of order $n$, along with a flow assignment $F = \{f_{ji}\}$, the flow balance of node $v_j$ is denoted by $b_j$ and is defined as $b_j = f_i^- - f_i^+$.

**Definition 3:** Given a digraph $G_d(V, \mathcal{E})$ of order $n$, along with a flow assignment $F = \{f_{ji}\}$, the absolute imbalance (or total imbalance) of digraph $G_d$ is denoted by $\varepsilon$ and is defined as $\varepsilon = \sum_{j=1}^{n} |b_j|$.

**Definition 4:** A digraph $G_d(V, \mathcal{E})$ of order $n$, along with a flow assignment $F = \{f_{ji}\}$, is called weight-balanced if its absolute imbalance (or total imbalance) is 0, i.e., $\varepsilon = \sum_{j=1}^{n} |b_j| = 0$.

With the definitions above, we can now formally state the flow assignment problem we are interested in solving in a distributed manner.

**Flow Assignment Problem:** We are given a strongly connected digraph $G_d(V, \mathcal{E})$, as well as lower and upper bounds $l_{ji}$ and $u_{ji}$ ($0 < l_{ji} \leq u_{ji}$) on each edge $(v_j, v_i) \in \mathcal{E}$. We want to develop a distributed algorithm that allows the nodes to iteratively adjust the flows on their outgoing edges so that they eventually obtain a set of flows $\{f_{ji} | (v_j, v_i) \in \mathcal{E}\}$ that satisfy the following properties:

**P1.** $0 < l_{ji} \leq f_{ji} \leq u_{ji}$ for each edge $(v_j, v_i) \in \mathcal{E}$; 

**P2.** $f_i^+ = f_i^-$ for every $v_j \in V$.

The distributed algorithm needs to respect the communication constraints imposed by the undirected graph $G_u$ that corresponds to the given digraph $G_d$.

If the necessary and sufficient conditions in the theorem below hold, obtaining a set of admissible flows (i.e., balanced and within the given constraints) can be obtained via a variety of centralized algorithms [2].

**Theorem 1:** (Circulation Theorem [2]) Consider a strongly connected digraph $G_d(V, \mathcal{E})$, with lower and upper bounds $l_{ji}$ and $u_{ji}$ ($0 < l_{ji} \leq u_{ji}$) on each edge $(v_j, v_i) \in \mathcal{E}$. The necessary and sufficient condition for the existence of a set of flows $\{f_{ji} | (v_j, v_i) \in \mathcal{E}\}$ that satisfy

1. **Interval constraints:** $0 < l_{ji} \leq f_{ji} \leq u_{ji}$, $\forall (v_j, v_i) \in \mathcal{E}$,
2. **Balance constraints:** $f_i^+ = f_i^-$, $\forall v_j \in V$,

is the following: for each $\mathcal{S}, \mathcal{S} \subset V$, we have

$$\sum_{(v_j,v_i) \in \mathcal{E}_\mathcal{S}} l_{ji} \leq \sum_{(v_j,v_i) \in \mathcal{E}_\mathcal{S}} u_{ji}$$  \hspace{1cm} (1)

where

$$\mathcal{E}_\mathcal{S} = \{(v_j,v_i) \in \mathcal{E} | v_j \in \mathcal{S}, v_i \in \mathcal{V} - \mathcal{S}\}, \hspace{1cm} (2)$$

$$\mathcal{E}_{\mathcal{S}}^\mathcal{S} = \{(v_j,v_i) \in \mathcal{E} | v_j \in \mathcal{S}, v_i \in \mathcal{V} - \mathcal{S}\}. \hspace{1cm} (3)$$

In the remainder of this paper, we assume that the above circulation conditions hold for a given directed graph and propose a distributed algorithm for allowing the nodes to find a flow assignment satisfying Properties P1 and P2. Additionally, we discuss an enhancement to the algorithm that allows the nodes to distributively determine whether or not the conditions in Theorem 1 hold.

### III. Distributed Flow Algorithm

In this section we first describe the distributed iterative algorithm that we propose to solve the constrained flow balancing problem. Then, we state the main results pertaining to the convergence of the algorithm.

#### A. Algorithm Description

Each node maintains and iteratively updates estimates of the flows on its incoming and outgoing edges so as to attempt to ensure, i.e., make the summation of the flow estimates on incoming edges equal to the sum of the flow estimates on outgoing edges. In the process, since any two connected nodes might be attempting to simultaneously adjust their flow estimate on the edge that connects them, they need to coordinate among themselves in order to reach an agreement on the flow estimate for that particular edge. Naturally, the nodes need to assign flows that respect the lower and upper limits on each edge; this, also needs to be taken into account by the nodes when adjusting the flow estimates they maintain.

Let $f_{ij}^{(j)}[k], v_i \in \mathcal{N}_j^-$, and $f_{ij}^{(j)}[k], v_i \in \mathcal{N}_j^+$, respectively, denote the flow estimates on incoming and outgoing edges maintain by node $v_j$ at iteration $k$. These estimates are updated by node $v_j$ based on its flow estimate imbalance at iteration $k$, denoted by $b_j^{(j)}[k]$, and defined as

$$b_j^{(j)}[k] := \sum_{v_i \in \mathcal{N}_j^-} f_{ij}^{(j)}[k] - \sum_{v_i \in \mathcal{N}_j^+} f_{ij}^{(j)}[k]. \hspace{1cm} (4)$$

**Initialization:** Initially, each node $v_j$ is aware of the feasible flow interval on each of its incoming and outgoing edges, i.e., node $v_j$ is aware of $l_{ij}, u_{ij}$ for each $v_i \in \mathcal{N}_j^-$ and $l_{ij}, u_{ij}$ for each $v_i \in \mathcal{N}_j^+$. Then, each node $v_j$ initializes the flow estimates it maintains at the middle of the feasible interval, i.e.,

$$f_{ij}^{(j)}[0] = \frac{l_{ij} + u_{ij}}{2},$$

$$f_{ij}^{(j)}[0] = \frac{l_{ij} + u_{ij}}{2}. \hspace{1cm} (5)$$

This initialization is not critical and could be any value in the feasible flow interval $[l_{ij}, u_{ij}]$.

**Iteration:** At each iteration $k \geq 0$, node $v_j$ updates the flow estimates on its incoming edges, $f_{ij}^{(j)}[k], v_i \in \mathcal{N}_j^-$, and outgoing edges, $f_{ij}^{(j)}[k], v_i \in \mathcal{N}_j^+$, by using Steps S1-S3 below.

[S1.] Each node $v_j$ attempts to change its flow estimates of both incoming and outgoing edges. The way this is done depends on whether or not the node has a positive flow estimate imbalance or not. We discuss both cases below and then describe how to concisely capture both.

(i) **Nodes with a positive flow estimate imbalance:** If $b_j^{(j)}[k] > 0$, node $v_j$ attempts to change the flow estimates it maintains for both its incoming edges $\{f_{ij}^{(j)}[k+1] \mid v_i \in \mathcal{N}_j^-\}$, and outgoing edges $\{f_{ij}^{(j)}[k+1] \mid v_i \in \mathcal{N}_j^+\}$ in a way that drives $b_j^{(j)}[k+1]$ to zero at the next iteration (at least if no other changes are inflicted on the flow estimates). More specifically, since node $v_j$ is associated with $D_j = D_j^- + D_j^+$ edges, it
attempts to change each incoming flow estimate by \(-\frac{b^{(j)}_i[k]}{D_j}\), and each outgoing flow estimate by \(+\frac{b^{(j)}_i[k]}{D_j}\), i.e., from the perspective of node \(v_j\), the desirable flow estimates at the next iteration are
\[
\hat{f}^{(j)}_j[k+1] = f^{(j)}_j[k] - \frac{b^{(j)}_j[k]}{D_j}, \quad v_i \in \mathcal{N}^-_j, \quad (6)
\]
\[
\hat{f}^{(j)}_j[k+1] = f^{(j)}_j[k] + \frac{b^{(j)}_j[k]}{D_j}, \quad v_i \in \mathcal{N}^+_j, \quad (7)
\]
where \(b_j[k] > 0\).

(ii) Nodes with a non-positive imbalance. If node \(v_j\) has \(b^{(j)}_j[k]\) that is negative or zero \((b^{(j)}_j[k] \leq 0)\), then node \(v_j\) does not attempt to make any flow changes.

Note that no desirable change on the flow estimates can also be captured by (6)–(7) with \(b^{(j)}_j[k] = 0\). Thus, regardless of whether node \(v_j\) has positive imbalance or not, we can capture the desirable new flow estimates on each incoming and outgoing edge as
\[
\hat{f}^{(j)}_j[k+1] = f^{(j)}_j[k] - \frac{b^{(j)}_j[k]}{D_j}, \quad v_i \in \mathcal{N}^-_j, \quad (8)
\]
\[
\hat{f}^{(j)}_j[k+1] = f^{(j)}_j[k] + \frac{b^{(j)}_j[k]}{D_j}, \quad v_i \in \mathcal{N}^+_j, \quad (9)
\]
where \(b^{(j)}_j[k]\) is defined as
\[
b^{(j)}_j[k] = \begin{cases} b^{(j)}_j[k], & \text{if } b^{(j)}_j[k] > 0, \\ 0, & \text{otherwise.} \end{cases}
\]

[S2.] Since the flow estimate \(f^{(j)}_j[k]\) on edge \((v_j, v_i) \in \mathcal{E}\) as maintained by \(v_j\) affects positively \(b^{(j)}_j[k]\) (maintained by node \(v_j\)), whereas the flow estimate \(f^{(j)}_j[k]\) on edge \((v_j, v_i) \in \mathcal{E}\) as maintained by \(v_i\) affects negatively \(b^{(j)}_i[k]\) (maintained by node \(v_i\)), we need to account for the possibility of both nodes attempting to inflict changes on their flow estimates. Thus, the new flow estimate on each edge \((v_j, v_i) \in \mathcal{E}\) as maintained by node \(v_j\) is taken to be
\[
f^{(j)}_j[k+1] = \frac{1}{2} \left( f^{(j)}_j[k+1] + \hat{f}^{(j)}_j[k+1] \right). \quad (10)
\]

[S3.] If the value in (10) is in interval \([l_{ji}, u_{ji}]\), then \(f^{(j)}_j[k+1] = f^{(j)}_j[k] + [l_{ji}, u_{ji}]\); otherwise, if it is above \(u_{ji}\) (respectively, below \(l_{ji}\)), it is set to the upper bound \(u_{ji}\) (respectively, to the lower bound \(l_{ji}\)):
\[
f^{(j)}_j[k+1] = \begin{cases} f^{(j)}_j[k+1], & \text{if } f^{(j)}_j[k+1] \leq u_{ji}, \\ u_{ji}, & \text{if } f^{(j)}_j[k+1] > u_{ji}, \\ l_{ji}, & \text{if } f^{(j)}_j[k+1] < l_{ji}. \end{cases} \quad (11)
\]

It follows from the initialization of the algorithm in (5), the averaging operation in (10), and the operation projection in (11), that \(f^{(j)}_j[k] = f^{(j)}_i[k], \forall k \geq 0\). [Note that even if \(f^{(j)}_i[k] \neq f^{(i)}_j[k]\), as long as \(f^{(j)}_i[k] \in [l_{ij}, u_{ij}]\) and \(f^{(i)}_i[k] \in [l_{ij}, u_{ij}]\), after Step S3, we will have that \(f^{(j)}_j[k] = f^{(j)}_i[k]\); thus, in this case, \(f^{(j)}_i[k] = f^{(i)}_j[k], \forall k \geq 0\).] Therefore, by defining \(f^{(j)}_j[k] := f^{(j)}_i[k] = f^{(j)}_i[k], \forall k \geq 0\), the progress of the algorithm in Steps S1-S3 can be summarized by a single iteration of the form
\[
f^{(j)}_j[k+1] = f^{(j)}_j[k] + \frac{\hat{b}_j[k]}{D_j} \left( \frac{b^{(j)}_i[k]}{D_i} - \frac{b^{(j)}_j[k]}{D_j} \right), \quad \forall (v_j, v_i) \in \mathcal{E}, \quad (12)
\]
where \([\cdot]_\mathbb{R}^\pm\) denotes the projection onto \([\mathbb{R}, \mathbb{R}^+\) and
\[
\hat{b}_j[k] = \begin{cases} b_j[k], & \text{if } b_j[k] > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (13)
\]
with
\[
b_j[k] = \sum_{v_i \in \mathcal{N}^-_j} f^{(j)}_j[k] - \sum_{v_i \in \mathcal{N}^+_j} f^{(j)}_i[k]. \quad (14)
\]

The compact description of the algorithm progress given in (12)–(14) greatly simplifies the notation and is convenient for convergence analysis purposes. It also highlights the fact that for distributed implementation purposes, each node \(v_j\) can transmit its \(b_j[k]/D_j\) at iteration \(k\) to all of its in- and out-neighbors, as opposed to having to transmit \(f^{(j)}_j[k+1]\) to

Algorithm 1: Distributed feasible flow algorithm

\begin{enumerate}
\item Each node \(v_j \in \mathcal{V}\) does the following:
\item Input: \(l_{ij}, u_{ij}, \forall v_i \in \mathcal{N}^+_j\)
\item Input: \(l_{ij}, u_{ij}, \forall v_i \in \mathcal{N}^-_j\)
\item Output: \(f^{(j)}_j[k], \forall v_i \in \mathcal{N}^+_j\)
\end{enumerate}

\begin{algorithm}
\begin{algorithmic}
\State Set \(f^{(j)}_i[k] = 0, \forall v_i \in \mathcal{N}^-_j\)
\State Set \(f^{(j)}_i[k] = 0, \forall v_i \in \mathcal{N}^+_j\)
\State Set \(D_j = D^-_j + D^+_j\)
\ForEach{iteration, \(k = 0, 1, \ldots\)}
\State Calculate: \(\hat{b}_j[k] = \sum_{v_i \in \mathcal{N}^-_j} f^{(j)}_i[k] - \sum_{v_i \in \mathcal{N}^+_j} f^{(j)}_i[k]\)
\State \(\hat{b}_j[k] = \begin{cases} \hat{b}_j[k], & \text{if } \hat{b}_j[k] > 0, \\ 0, & \text{otherwise.} \end{cases} \)
\State Transmit: \(\hat{b}_j[k]/D_j\) to \(v_i \in \mathcal{N}^-_j\) and \(v_i \in \mathcal{N}^+_j\)
\State Receive: \(\hat{b}_j[k]/D_i\) from all \(v_i \in \mathcal{N}^-_j\) and \(\hat{b}_j[k]/D_i\) from all \(v_i \in \mathcal{N}^+_j\)
\State Calculate:
\State \(f^{(j)}_i[k+1] = f^{(j)}_i[k] + \frac{1}{2} \left( b^{(j)}_j[k] - b^{(j)}_i[k] \right), v_i \in \mathcal{N}^-_j\)
\State \(f^{(j)}_i[k+1] = f^{(j)}_i[k] + \frac{1}{2} \left( b^{(j)}_j[k] - b^{(j)}_i[k] \right), v_i \in \mathcal{N}^+_j\)
\State Set:
\State \(f^{(j)}_i[k+1] = \begin{cases} f^{(j)}_i[k+1], & \text{if } f^{(j)}_i[k+1] \in [l_{ij}, u_{ij}], \\ u_{ij}, & \text{if } f^{(j)}_i[k+1] > u_{ij}, \\ l_{ij}, & \text{if } f^{(j)}_i[k+1] < l_{ij}. \end{cases} \)
\State \(f^{(j)}_i[k+1] = \begin{cases} f^{(j)}_i[k+1], & \text{if } f^{(j)}_i[k+1] \in [l_{ij}, u_{ij}], \\ u_{ij}, & \text{if } f^{(j)}_i[k+1] > u_{ij}, \\ l_{ij}, & \text{if } f^{(j)}_i[k+1] < l_{ij}. \end{cases} \)
\EndFor
\end{algorithmic}
\end{algorithm}
that nodes with a positive flow balance retain a positive flow expected (this is established in the next section). We observe \( \varepsilon \) In the middle of Fig. 2, we plot the evolution of the absolute zero (this is also something we establish in the next section). Notice that the sum \( k \) of the distributed flow balancing algorithm (Algorithm 1). On the right of Fig. 2, we plot the values of the flows \( c \) for each \( v_j, v_i \in \mathcal{E} \). It can be observed that the flows eventually stabilize to fixed values, given by the matrix of flows \( f_{ji}[k] \) for each \( (v_j, v_i) \in \mathcal{E} \). which are easily seen to be feasible and result in a balanced digraph. □

**Example 1:** In this example, we illustrate the operation of Algorithm 1 for a randomly generated (strongly connected) digraph with seven nodes. The adjacency matrix \( A \) for the digraph is shown in Fig. 1, along with the lower and upper bounds for the edges in this graph. [Recall that the adjacency matrix \( A \) of a digraph \( \mathcal{G}_d \) has \( A(j, i) = 1 \) if \( (v_j, v_i) \in \mathcal{E} \), otherwise \( A(j, i) = 0 \).] It can be verified that the circulation conditions in Theorem 1 are satisfied.

On the left of Fig. 2, we plot the flow balance, \( b_j[k] \), \( j = 1, 2, ..., 7 \), of each of the seven nodes against the iteration \( k \) of the distributed flow balancing algorithm (Algorithm 1). Notice that the sum \( \sum_{j=1}^{7} b_j[k] \) is identically zero for all \( k \) as expected (this is established in the next section). We observe that nodes with a positive flow balance retain a positive flow balance as \( k \) increases; in the end, only one node retains a negative balance, and all node balances asymptotically go to zero (this is also something we establish in the next section). In the middle of Fig. 2, we plot the evolution of the absolute (total) imbalance \( \varepsilon[k] \) against the iteration \( k \). Notice that \( \varepsilon[k] \) monotonically goes to zero (again, this is a key result in our proof of convergence in the next section).

On the right of Fig. 2, we plot the values of the flows \( f_{ji}[k] \) for each \( (v_j, v_i) \in \mathcal{E} \). It can be observed that the flows eventually stabilize to fixed values, given by the matrix of flows

\[
F = \begin{bmatrix}
0 & 4.8848 & 5.4988 & 0 & 0 & 1 & 7.9926 \\
5.6152 & 0 & 0 & 0 & 1 & 0 & 8.6078 \\
7.0012 & 0 & 0 & 0 & 2 & 3.7488 & 0 \\
0 & 2.9461 & 0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 9 & 0 \\
4.7525 & 4 & 7.2512 & 0 & 5 & 0 & 0 \\
2.0074 & 3.3922 & 0 & 6.9461 & 0 & 4.2549 & 0 \\
\end{bmatrix}
\]

which are easily seen to be feasible and result in a balanced digraph.

**B. Convergence Results**

Next, we state the main convergence results of the paper; namely that Algorithm 1 converges to a set of feasible and balanced flows, as long as the necessary and sufficient condition in (1) holds. In particular, the absolute (total) imbalance \( \varepsilon[k] \) in Definition 3 goes to zero as \( k \) goes to infinity, with a geometric rate. This in turn implies that the flow balance \( b_j[k] \) for each node \( v_j \in \mathcal{V} \) goes to zero; therefore, \( f_{ji}[k] = f_{ji}^{(j)}[k] = f_{ji}^{(i)}[k] \), \( \forall (v_j, v_i) \in \mathcal{E} \), converge to \( f_{ji}^{*} \), \( \forall (v_j, v_i) \in \mathcal{E} \), as \( k \) goes to infinity, such that the converged flows are within the given lower and upper limits, i.e., \( l_{ji} \leq f_{ji}^{*} \leq u_{ji} \), \( \forall (v_j, v_i) \in \mathcal{E} \). The proofs of these results is deferred to Sections IV and V.

**Theorem 2:** Consider a strongly connected digraph \( \mathcal{G}_d = (\mathcal{V}, \mathcal{E}) \) of order \( n \geq 2 \), with lower and upper bounds \( l_{ji} \) and \( u_{ji} \) \( (0 < l_{ji} \leq u_{ji}) \) on each edge \( (v_j, v_i) \in \mathcal{E} \), such that the necessary and sufficient condition in (1) holds. During the
execution of (12)--(14), with the initial conditions in (5), it holds that

\[ \varepsilon[k + n] \leq (1 - c)\varepsilon[k], \quad \forall k \geq 0, \]

where \( \varepsilon[k] \geq 0 \) is the absolute (total) imbalance of the network at iteration \( k \) (refer to Definition 3), with

\[ c = \frac{1}{2n} \left( \frac{1}{2D_{\text{max}}} \right)^n, \]

where \( D_{\text{max}} = \max_{v_j \in V} D_j \). [Note that \( D_{\text{max}} \) necessarily satisfies 1 \( \leq D_{\text{max}} \leq 2(n - 1) \].

**Corollary 1:** Consider a strongly connected digraph \( G_d = (V, E) \) of order \( n \geq 2 \), with lower and upper bounds \( l_{ji} \) and \( u_{ji} \) \((0 < l_{ji} \leq u_{ji})\) on each edge \((v_j, v_i) \in E\), such that the necessary and sufficient condition in (1) holds. The execution of (12)--(14), with the initial conditions in (5) asymptotically leads to a set of flows \( \{f^k_{ji} \mid (v_j, v_i) \in E\} \) that satisfy the interval constraints and balance constraints, i.e., we have \( \lim_{k \to \infty} f^k_{ji}[k] = f^*_{ji} \), \( \forall (v_j, v_i) \in E \), where the set of flows \( \{f^k_{ji} \mid (v_j, v_i) \in E\} \) satisfy

1) \( l_{ji} \leq f^k_{ji}[k] \leq u_{ji}, \forall (v_j, v_i) \in E; \)
2) \( \sum_{v_i \in N^-_j} f^k_{ji} = \sum_{v_i \in N^+_j} f^k_{ij}, \forall v_j \in V. \)

**IV. Ancillary Results**

In this section, we establish several results that are utilized in Section V for proving the main result in Theorem 2. For easing subsequent developments, we make use of the following notation.

**Definition 5:** The flow change incurred at the flow of edge \((v_j, v_i) \in E\) at iteration \( k \) is denoted by \( \Delta f^k_{ji}[k] \), i.e.,

\[ \Delta f^k_{ji}[k] := f^k_{ji}[k + 1] - f^k_{ji}[k], \]  

a variable that captures the combined effect of both (10) and (11). Similarly, we define the changes in the flow balances at each node \( v_j \in V \) and the absolute (total) imbalance of the network:

\[ \Delta b^k_j[k] \equiv b^k_j[k + 1] - b^k_j[k], \quad \Delta \varepsilon^k[k] \equiv \varepsilon^k[k + 1] - \varepsilon^k[k]. \]

The following proposition establishes some basic relationships governing the flow node balances of sets of nodes and the absolute (total) imbalance of the network.

**Proposition 1:** Consider a strongly connected digraph \( G_d = (V, E) \) of order \( n \geq 2 \), with lower and upper bounds \( l_{ji} \) and \( u_{ji} \) \((0 < l_{ji} \leq u_{ji})\) on each edge \((v_j, v_i) \in E\), such that the necessary and sufficient condition in (1) holds. At each iteration \( k \) during the execution of (12)--(14), with the initial conditions in (5), it holds that

1) For any subset of nodes \( S \subset V \), if we let \( \mathcal{E}^-_S \) and \( \mathcal{E}^+_S \) be defined by (2) and (3), respectively, we have

\[ \sum_{v_j \in S} b^k_j[k] = \sum_{(v_j, v_i) \in \mathcal{E}^-_S} f^k_{ji}[k] = \sum_{(v_j, v_i) \in \mathcal{E}^+_S} f^k_{ij}[k]; \]
2) \( \sum_{j=1}^n b^k_j[k] = 0; \)
3) \( \varepsilon^k[k] = 2 \sum_{v_j \in V^+[k]} b^k_j[k] \) where \( V^+[k] = \{v_j \in V \mid b^k_j[k] > 0\} \).

**Proof 1:** To prove the first statement, let

\[ \mathcal{E}_S = \{(v_j, v_i) \in E \mid v_j \in S, v_i \in S\} \]

be the set of edges that are internal to the set \( S \). From the definition of the flow balance for node \( v_j \), we have (after rearranging the summations)

\[ \sum_{v_j \in S} b^k_j[k] = \sum_{v_j \in S} \left( \sum_{v_i \in N^-_j} f^k_{ji}[k] - \sum_{v_i \in N^+_j} f^k_{ij}[k] \right) \]
\[ = \sum_{(v_j, v_i) \in \mathcal{E}^-_S} f^k_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}^+_S} f^k_{ij}[k] + \sum_{(v_j, v_i) \in \mathcal{E}^-_S} f^k_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}^+_S} f^k_{ij}[k] \]
\[ = \sum_{(v_j, v_i) \in \mathcal{E}^-_S} f^k_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}^+_S} f^k_{ij}[k]. \]

For the second statement, we can take any \( S \subset V \) and argue

\[ \sum_{v_j \in V} b^k_j[k] = \sum_{v_j \in S} b^k_j[k] + \sum_{v_j \in V \setminus S} b^k_j[k] = 0. \]

For the third statement, notice that, from the definition of \( \varepsilon^k[k] \) in Definition 3, we have

\[ \varepsilon^k[k] = \sum_{v_j \in V} |b^k_j[k]| = \sum_{v_j \in V^+[k]} |b^k_j[k]| + \sum_{v_j \in V \setminus V^+[k]} |b^k_j[k]| \]
\[ = \sum_{v_j \in V^+[k]} b^k_j[k] - \sum_{v_j \in V \setminus V^+[k]} b^k_j[k] = 2 \sum_{v_j \in V^+[k]} b^k_j[k], \]

utilizing the definition of \( V^+[k] \) (all nodes have positive balance) and the second statement of this proposition. \( \square \)

The following proposition establishes certain monotonic properties governing the nodes that have positive flow balance.

**Proposition 2:** Consider a strongly connected digraph \( G_d = (V, E) \) of order \( n \geq 2 \), with lower and upper bounds \( l_{ji} \) and \( u_{ji} \) \((0 < l_{ji} \leq u_{ji})\) on each edge \((v_j, v_i) \in E\), such that the necessary and sufficient condition in (1) holds. During the execution of (12)--(14), with the initial conditions in (5), we have the following:

1) \( b^k_j[k + 1] \geq \frac{1}{2} b^k_j[k] > 0, \quad \forall v_j \in V^+[k]; \)
2) \( V^+[k] \subset V^+[k + 1]. \)

**Proof 2:** Consider a node \( v_j \in V^+[k] \) with balance \( b^k_j[k] > 0. \) Suppose for now that (i) all neighbors of node \( v_j \) do not belong in \( V^+[k] \) (i.e., \( N^-_j \cap V^+[k] = \emptyset \)) and that (ii) during the update of the flows (following (10)), the flows on each edge of node \( v_j \) are within the lower and upper limits. Then, since nodes outside the set \( V^+[k] \) posses a \( b \) of zero, it follows that the flows are updated as

\[ f^k_{ji} = f^k_{ji} - \frac{b^k_j[k]}{2D_j}, \forall v_i \in N^-_j, \]
\[ f^k_{ij} = f^k_{ij} + \frac{b^k_j[k]}{2D_j}, \forall v_i \in N^+_j. \]

Thus, the flow balance of node \( v_j \) satisfies

\[ b^k_j[k + 1] = \sum_{v_i \in N^-_j} f^k_{ji}[k + 1] - \sum_{v_i \in N^+_j} f^k_{ij}[k + 1] = -\frac{D_j^{-1}b^k_j[k]}{2D_j} + \frac{b^k_j[k]}{2D_j} \]
\[ = \frac{b^k_j[k]}{2D_j} + b^k_j[k] = \frac{1}{2} b^k_j[k]. \]
For the general case (when node \( v_j \) may have neighbors with positive flow balance and/or the flows reach the lower or upper flow limits on the corresponding edges), we make the following observations:

1. If an in-neighbor \( v_i \in \mathcal{N}_j^- \) has positive flow balance, the flow \( f_{ji} \) will satisfy

\[
 f_{ji}[k+1] = f_{ji}[k] - \frac{b_j[k]}{2D_j} , \forall v_i \in \mathcal{N}_j^- .
\]

To see this, notice that:

(i) If \( f_{ji}[k+1] > u_{ji} \), then \( f_{ji}[k+1] = u_{ji} \)

(ii) If \( f_{ji}[k+1] < l_{ji} \), then \( f_{ji}[k+1] = l_{ji} \).

2. Similar arguments (but reversed) can be used to establish that if an out-neighbor \( v_i \in \mathcal{N}_j^+ \) has positive flow balance, the flow \( f_{ij} \) will satisfy

\[
 f_{ij}[k+1] \leq f_{ij}[k] + \frac{b_j[k]}{2D_j} , \forall v_i \in \mathcal{N}_j^+ .
\]

Thus, we conclude that in the general case we still have

\[
 b_j[k+1] = -\sum_{v_i \in \mathcal{N}_j^-} (f_{ji}[k] - f_{ji}[k]) + b_j[k] = -\sum_{v_i \in \mathcal{N}_j^-} b_j[k] .
\]

The second statement in the proposition follows trivially from the first: a positive node remains positive; thus, the set \( \mathcal{V}^+ \) can only be enlarged.

The following proposition establishes that the absolute (total) imbalance of the network is monotonically non-increasing.

**Proposition 3:** Consider a strongly connected digraph \( G_d = (\mathcal{V}, \mathcal{E}) \) of order \( n \) greater, with lower and upper bounds \( l_{ji} \) and \( u_{ji} \) on each edge \( (v_j, v_i) \in \mathcal{E} \). Assume that the necessary condition (1) holds.

In the execution of (12)–(14), with the initial conditions in (5), it has that \( 0 \leq \varepsilon[k+1] \leq \varepsilon[k] \).

**Proof 3:** Let \( \mathcal{V}^+ \) be the set of nodes with positive flow balance at iteration \( k \), i.e., \( \mathcal{V}^+ = \{ v_j \in \mathcal{V} \mid b_j[k] > 0 \} \) (note that \( \mathcal{V}^+ \) has to be a strict subset of \( \mathcal{V} \), e.g., because of the second statement of Proposition 1). Taking \( \mathcal{S} = \mathcal{V}^+ \subseteq \mathcal{V} \), define \( \mathcal{E}_\mathcal{S}^- \) and \( \mathcal{E}_\mathcal{S}^+ \) as in (2) and (3) respectively, and let \( \mathcal{E}_\mathcal{S} = \{(v_j, v_i) \in \mathcal{E} \mid v_j \in \mathcal{S}, v_i \in \mathcal{S} \} \) be the set of edges that are internal to the set \( \mathcal{S} \). Note that \( \Delta f_{ji} \) necessarily satisfies (e.g., see the discussion in the beginning of the proof of Proposition 2): \( \Delta f_{ji} \leq 0, \forall (v_j, v_i) \in \mathcal{E}_\mathcal{S}^- \), \( \Delta f_{ji} \leq 0, \forall (v_j, v_i) \in \mathcal{E}_\mathcal{S}^+ \), \( \Delta f_{ji} = 0, \forall (v_j, v_i) \in \mathcal{E} - (\mathcal{E}_\mathcal{S}^- \cup \mathcal{E}_\mathcal{S}^+ \cup \mathcal{E}_\mathcal{S}) \).

Since all nodes in \( \mathcal{S} = \mathcal{V}^+ \) have positive flow balance at iteration \( k \) and all remaining nodes have non-positive flow balance at iteration \( k \), we have

\[
 \varepsilon[k] = \sum_{v_j \in \mathcal{V}} |b_j[k]| = \sum_{v_j \in \mathcal{S}} b_j[k] + \sum_{v_j \in \mathcal{V}^+} |b_j[k]| = \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} f_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} f_{ji}[k] + \sum_{v_j \in \mathcal{V}^+} |b_j[k]| ,
\]

where in the last line we utilized the result of the first statement of Proposition 1.

Similarly, since from Proposition 2, it holds \( \mathcal{S} = \mathcal{V}^+ \subseteq \mathcal{V}^+ \), we have

\[
 \varepsilon[k+1] = \sum_{v_j \in \mathcal{S}} b_j[k+1] + \sum_{v_j \in \mathcal{V}^+} |b_j[k+1]| = \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} f_{ji}[k+1] - \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} f_{ji}[k+1] + \sum_{v_j \in \mathcal{V}^+} |b_j[k+1]| .
\]

Putting the two above equations together, we obtain:

\[
 \Delta \varepsilon[k] = \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} \Delta f_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} \Delta f_{ji}[k] + \sum_{v_j \in \mathcal{V}^+} \left( |b_j[k+1]| - |b_j[k]| \right) .
\]

For \( v_j \in \mathcal{V}^+ \) we have

\[
 b_j[k+1] = \sum_{v_i \in \mathcal{N}_j^-} f_{ji}[k+1] = \sum_{v_i \in \mathcal{N}_j^+} f_{ji}[k+1] + \sum_{v_i \in \mathcal{N}_j} f_{ji}[k] + \Delta f_{ji}[k] = b_j[k] + \sum_{v_i \in \mathcal{N}_j} \Delta f_{ji}[k] - \sum_{v_i \in \mathcal{N}_j} \Delta f_{ij}[k] ,
\]

and by the triangle inequality

\[
 |b_j[k+1]| \leq |b_j[k]| + \sum_{v_i \in \mathcal{N}_j^-} |\Delta f_{ji}[k]| + \sum_{v_i \in \mathcal{N}_j^+} |\Delta f_{ij}[k]| .
\]

Thus the last term in (16) satisfies

\[
 \sum_{v_j \in \mathcal{V}^+} \left( |b_j[k+1]| - |b_j[k]| \right) \leq \sum_{v_j \in \mathcal{V}^+} \left( \sum_{v_i \in \mathcal{N}_j^-} |\Delta f_{ji}[k]| + \sum_{v_i \in \mathcal{N}_j^+} |\Delta f_{ij}[k]| \right) \leq \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} |\Delta f_{ji}[k]| + \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} |\Delta f_{ij}[k]| ,
\]

where the last line follows from the fact that the edges, the flows of which change and are incident with nodes in the set \( \mathcal{V}^+ - \mathcal{S} \), are in the edges in \( \mathcal{E}_\mathcal{S}^- \) and \( \mathcal{E}_\mathcal{S}^+ \) (the other flows in question do not change).

Going back to (16), we have that

\[
 \Delta \varepsilon[k] = \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} \Delta f_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} \Delta f_{ji}[k] + \sum_{v_j \in \mathcal{V}^+} \left( |b_j[k+1]| - |b_j[k]| \right) \leq \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} \Delta f_{ji}[k] - \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} \Delta f_{ij}[k] + \sum_{v_j \in \mathcal{V}^+} \left( |\Delta f_{ji}[k]| + \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^-} |\Delta f_{ji}[k]| + \sum_{(v_j, v_i) \in \mathcal{E}_\mathcal{S}^+} |\Delta f_{ij}[k]| \right) = 0 .
\]
where the last line follows form the fact that $\Delta f_{j_1}[k] \leq 0$ 

$\forall (v_j, v_i) \in E^-_S$, and $\Delta f_{j_1}[k] \geq 0$, $\forall (v_i, v_j) \in E^+_S$. □

The following proposition is a refinement of Proposition 3. It basically states that at each iteration $k$ of the algorithm described by (5)–(11), the change in the absolute (total) imbalance of the network depends exclusively on (i) the changes in flows on edges that connect positive nodes in $V^+ \setminus S$ and negative nodes in $V^- \setminus V^+\setminus S$, and (ii) the changes in the flow balances of nodes with non-positive balance that are directly connected with nodes with positive flow balance.

**Proposition 4:** Consider a strongly connected directed graph $G_d = (V, E)$ of order $n \geq 2$, with lower and upper bounds $l_{ij}$ and $u_{ij}$ on each edge $(v_j, v_i) \in E$, such that the necessary and sufficient condition in (1) holds. At time step $k$ of the execution of (12)–(14), with the initial conditions in (5), let $S \subset V^+ \cap V^- \subset V$ be the set of nodes with positive flow balance at iteration $k$ (i.e., $V^+ \setminus S = \{ v_j \in V \mid b_j[k] > 0 \}$) and let $S = V \setminus S$ be the remaining nodes (with zero or negative flow balance). Define $E^-_S$ and $E^+_S$ as in (2) and (3) respectively, and let $T \subseteq S$ be the subset of nodes in $S$ directly connected to nodes in $S$ (i.e., $T = \{ v_j \in S \mid \exists v_i \text{ s.t.} (v_i, v_j) \in E^-_S \text{ or } (v_j, v_i) \in E^+_S \}$). We have

$$\Delta \varepsilon[k] = 2 \left( \sum_{(v_j, v_i) \in E^-_S} \Delta f_{j_1}[k] \right) + \sum_{v_j \in T} (b_j[k + 1] + b_j[k + 1]) \quad (17)$$

$$= \sum_{v_j \in T} \Delta \varepsilon_j[k] , \quad (18)$$

where

$$\Delta \varepsilon_j[k] = |b_j[k + 1]| + b_j[k + 1] + \sum_{v_i \in N^-_j \cap S} 2 \Delta f_{j_1}[k] + \sum_{v_i \in N^+_j \cap S} 2 \Delta f_{j_1}[k] \leq 0 . \quad (19)$$

**Proof 4:** From (16) in the proof of Proposition 3, we have

$$\Delta \varepsilon[k] = \sum_{(v_j, v_i) \in E^-_S} \Delta f_{j_1}[k] - \sum_{(v_i, v_j) \in E^+_S} \Delta f_{j_1}[k] + \sum_{v_j \in S} (|b_j[k + 1]| + b_j[k]) \quad (20)$$

$$= \sum_{(v_j, v_i) \in E^-_S} \Delta f_{j_1}[k] - \sum_{(v_i, v_j) \in E^+_S} \Delta f_{j_1}[k] + \sum_{v_j \in S} (|b_j[k + 1]| + b_j[k + 1]) \quad (21)$$

$$- \sum_{v_j \in S} (b_j[k + 1] + b_j[k]) , \quad (22)$$

where we used the fact that $b_j[k] \leq 0$ for $v_j \in S$. Furthermore, we have

$$- \sum_{v_j \in S} (b_j[k + 1] + b_j[k])$$

$$= \sum_{v_j \in S} (b_j[k + 1] - b_j[k])$$

$$= \sum_{(v_j, v_i) \in E^-_S} \Delta f_{j_1}[k] - \sum_{(v_i, v_j) \in E^+_S} \Delta f_{j_1}[k] ,$$

where the second (third) line follows from Statement 2 (Statement 1) in Proposition 1 and the definition of $\Delta f_{j_1}[k]$. To arrive at (17), we realize that the term $|b_j[k + 1]| + b_j[k + 1]$ is necessarily zero for all nodes in the set $S \setminus T$: the reason is that the flows on edges incident to these nodes do not change at all; thus, $b_j[k + 1] = b_j[k] \leq 0$ and therefore $|b_j[k + 1]| = -b_j[k + 1]$. To complete the proof, we reorder the two summations of the flow changes in the set of edges in $E^-_S$ ($E^+_S$) in terms of outgoing (incoming) edges of nodes in the set $T$, which leads us to (18). The fact that $\Delta \varepsilon_j[k] \leq 0$ can be seen as follows: (i) If $b_j[k + 1] \leq 0$, then $|b_j[k + 1]| + b_j[k + 1] = 0$; thus, since $\Delta f_{j_1}[k] \geq 0$ for all $v_j \in T$ and $v_i \in N^-_j \cap S$, and $\Delta f_{j_1}[k] \leq 0$ for all $v_j \in T$ and $v_i \in N^+_j \cap S$, we have $\Delta \varepsilon_j[k] \leq 0$. (ii) If $b_j[k + 1] \geq 0$, then since $b_j[k + 1] - b_j[k] = +\sum_{v_i \in N^-_j \cap S} \Delta f_{j_1}[k] - \sum_{v_i \in N^+_j \cap S} \Delta f_{j_1}[k]$ and $b_j[k] \leq 0$, we have

$$0 \leq b_j[k + 1] \leq + \sum_{v_i \in N^-_j \cap S} \Delta f_{j_1}[k] - \sum_{v_i \in N^+_j \cap S} \Delta f_{j_1}[k] .$$

Thus, $|b_j[k + 1]| + b_j[k + 1] = 2b_j[k + 1]$ and

$$2b_j[k + 1] \leq + \sum_{v_i \in N^-_j \cap S} \Delta f_{j_1}[k] - \sum_{v_i \in N^+_j \cap S} \Delta f_{j_1}[k] ;$$

thus, $\Delta \varepsilon_j[k] \leq 0$. □

V. PROOF OF MAIN CONVERGENCE RESULTS

We are now ready to move with the proof of Theorem 2.

**Proof of Theorem 2:** Let $v_{j_{\text{max}}} \in V^+ [k]$ be the node with the maximum (positive) flow balance at iteration $k$. It follows from the third statement of Proposition 1 that $b_{j_{\text{max}}}[k] \geq \frac{|\varepsilon[k]|}{2^{\sum \Delta f_{j_1}[k]}} \geq \frac{|\varepsilon[k]|}{2n}$ (a tighter lower bound would have been $|\varepsilon[k]|/(2(n-1))$ but it is more convenient to use the above); therefore, for all $t = 0, 1, 2, \ldots$, we have (from the first statement of Proposition 2)

$$b_{j_{\text{max}}}[k + t] \geq \left( \frac{1}{2} \right)^t \frac{|\varepsilon[k]|}{2n} \geq \left( \frac{1}{2D_{\text{max}}} \right)^t \frac{|\varepsilon[k]|}{2n} .$$

Note that there also exists a node $v_{j_{\text{min}}}$ with the minimum (negative) flow balance at iteration $k$, the flow balance of which satisfies $b_{j_{\text{min}}}[k] \leq -\frac{|\varepsilon[k]|}{2^{\sum \Delta f_{j_1}[k]}} \leq -\frac{|\varepsilon[k]|}{2n}$.

We recursively define the sets of nodes $V_k, V_{k + 1}, V_{k + 2}, \ldots, V_{k + n - 1}$, all of which are subsets of $V$:

1. $V_k = \{ v_{j_{\text{max}}} \}$
2. For $t = 1, 2, \ldots, n - 2$, we let

$$V_{k + t} = V_{k + t - 1} \cup V^+_{k + t - 1} \cup V^-_{k + t - 1}$$

where

$$V^+_{k + t - 1} = \{ v_j \in V \mid \exists v_i \in V_{k + t - 1} \text{ s.t. } (v_i, v_j) \in \mathcal{E} \text{ and } f_{j_1}[k + t] \leq u_{ij} \} ,$$

$$V^-_{k + t - 1} = \{ v_j \in V \mid \exists v_i \in V_{k + t - 1} \text{ s.t. } (v_i, v_j) \in \mathcal{E} \text{ and } f_{j_1}[k + t] \geq l_{ij} \} .$$

For $t = 0, 1, 2, \ldots, n - 1$, consider the inequality

$$\left( \frac{1}{2D_{\text{max}}} \right)^t \frac{|\varepsilon[k]|}{2n} - g[t] > 0 , \quad (20)$$

where \( g(t) = \varepsilon[k] - \varepsilon[k + t] \geq 0 \) is the gain in the absolute (total) imbalance after \( t \) iterations. Note that if the above inequality is violated at some \( t_0 \in \{1, 2, ..., n - 1\} \) (without loss of generality, let \( t_0 \) be the smallest such integer when the inequality is violated for the first time), then we have

\[
g[t_0] \geq \left( \frac{1}{2D_{\text{max}}} \right)^{t_0} \frac{\varepsilon[k]}{2n},
\]

which implies that

\[
\varepsilon[k + t_0] - \varepsilon[k] \leq \varepsilon[k] - \left( \frac{1}{2D_{\text{max}}} \right)^{t_0} \frac{\varepsilon[k]}{2n} \leq \varepsilon[k] \left( 1 - \frac{1}{2n} \left( \frac{1}{2D_{\text{max}}} \right)^{t_0} \right) \varepsilon[k],
\]

(21)

which immediately leads to the proof of the theorem (since, by Proposition 3, \( \varepsilon[k + n] \leq \varepsilon[k + t_0] \) for \( n \geq t_0 \).

We will argue, by contradiction, that the inequality in (20) gets violated for the first time at some \( t_0 \in \{0, 1, 2, ..., n - 1\} \), which will establish our proof. Suppose that the inequality (20) holds for all \( t \in \{0, 1, 2, ..., n - 1\} \). Then, we argue below that each node \( v_j \) in the set \( V_{k+t} \), \( t \in \{0, 1, 2, ..., n - 1\} \), has flow balance that satisfies

\[
b_j[k + t] \geq \left( \frac{1}{2D_{\text{max}}} \right)^{t} \frac{\varepsilon[k]}{2n} - g[t] > 0.
\]

(23)

This is established at the end of the proof. Assuming (for now) that (23) holds, we have

\[
\sum_{v_j \in V_{k+t}} b_j[k + t] > 0, \forall t \in \{0, 1, ..., n - 1\}.
\]

(24)

Since, by construction, we have

\[
V_k \subseteq V_{k+1} \subseteq V_{k+2} \subseteq \ldots \subseteq V_{k+n-1} \subseteq V
\]

and \( |V| = n \), we need to have \( V_{k+t} = V_{k+t-1} \) for some \( t \in \{1, 2, ..., n - 1\} \). Then, we have two possibilities, both of which lead to a contradiction:

(1) \( V_{k+1} = V \), which immediately leads to a contradiction in (24) (because \( \sum_{v_j \in V} b_j[k] = 0 \) for all \( k \) by the second statement of Proposition 1).

(2) If \( V_{k+t} \subset V \), let \( S = V_{k+t} = V_{k+t-1} \) and define \( E^-_S \) and \( E^+_S \) as in (2) and (3) respectively. Then, from the recursive definition of \( V_{k+t} \) we have

\[
f_{j}[k + t] = l_{j}, \forall (v_j, v_i) \in E^-_S,
\]

\[
f_{ij}[k + t] = u_{ij}, \forall (v_i, v_j) \in E^+_S.
\]

[Note that both \( E^+_S \) and \( E^-_S \) are nonempty sets; otherwise, the given graph \( G_d = (V, E) \) would not be strongly connected. Furthermore, if the upper (respectively, lower) limits were not reached for edges in \( E^+_S \) (respectively, \( E^-_S \)), the set \( V_{k+t} \) would strictly contain \( V_{k+t-1} \).] Thus, from the first statement of Proposition 1, we have

\[
\sum_{v_j \in S} b_j[k + t] = \sum_{(v_j, v_i) \in E^-_S} f_{ji}[k + t] - \sum_{(v_i, v_j) \in E^+_S} f_{ij}[k + t] = \sum_{(v_j, v_i) \in E^-_S} l_{ji} - \sum_{(v_i, v_j) \in E^+_S} u_{ij}.
\]

Since, all nodes in \( V_{k+t} \) have strictly positive balance, we have

\[
\sum_{(v_j, v_i) \in E^-_S} l_{ji} - \sum_{(v_i, v_j) \in E^+_S} u_{ij} > 0,
\]

which contradicts the circulation conditions in Theorem 1.

We now argue that if the inequality (20) holds for \( t \in \{0, 1, ..., n - 1\} \), then inequality (23) also holds for \( t \in \{0, 1, ..., n - 1\} \). The proof is by induction. Clearly, the inequality holds for \( t = 0 \). Suppose that (23) holds for \( k + t \), i.e., for all \( v_j \in V_{k+t} \), we have

\[
b_j[k + t] \geq \left( \frac{1}{2D_{\text{max}}} \right)^{t} \frac{\varepsilon[k]}{2n} - g[t] > 0,
\]

where \( g[t] = \varepsilon[k] - \varepsilon[k + t] \) is the gain in the absolute balance after \( t \) iterations. We need to argue that

\[
b_j[k + t + 1] \geq \left( \frac{1}{2D_{\text{max}}} \right)^{t+1} \frac{\varepsilon[k]}{2n} - g[t + 1] > 0
\]

for all \( v_j \in V_{k+t+1} \).

Since \( g[t + 1] \geq g[t] \geq 0 \) (follows from Proposition 3) and \( b_j[k + 1] \geq \frac{1}{2} b_j[k] \) for nodes with positive flow balance (from the first statement of Proposition 2), the above trivially holds for nodes in the set \( V_{k+t} \) (which necessarily belong in the set \( V_{k+t+1} \)). Let us now consider nodes in the set \( V_{k+t+1} \), which (by construction of the set \( V_{k+t+1} \)) have to necessarily share edges with nodes in the set \( V_{k+t} \). Thus, we consider three possibilities:

Case 1: \( v_j \in V_{k+t+1} \setminus V_{k+t} \), such that there exists at least one \( v_i \in V_{k+t} \) with \( (v_i, v_j) \in E \) and \( f_{ij}[k + t + 1] \leq u_{ij} \);

Case 2: \( v_i \in V_{k+t+1} \setminus V_{k+t} \), such that there exists at least one \( v_j \in V_{k+t} \) with \( (v_i, v_j) \in E \) and \( f_{ji}[k + t + 1] \geq l_{ij} \);

Case 3: A combination of the above two cases.

We focus on Case 1 since Cases 2 and 3 can be treated similarly. We have two possibilities to consider: (i) \( b_i[k + t] > 0 \) and (ii) \( b_i[k + t] \leq 0 \).

(i) If \( b_i[k + t] > 0 \), then \( f_{ji}[k + t + 1] = f_{ji}[k + t] + \frac{b_j[k + t]}{2D_j} - \frac{b_i[k + t]}{2D_i} \) and, since \( f_{ji}[k + t + 1] \leq u_{ij} \) (by construction of \( V_{k+t+1} \)), we have \( f_{ij}[k + t + 1] \geq f_{ij}[k + t] + \frac{b_j[k + t]}{2D_j} - \frac{b_i[k + t]}{2D_i} \).

Consider the flow balance of node \( v_i \) at iteration \( k + t + 1 \). Using an argument similar to the proof of the first statement of Proposition 2, but also taking into account the flow \( f_{ij}[k + t + 1] \), we have

\[
b_i[k + t + 1] \geq \frac{1}{2} b_i[k + t] + \frac{b_j[k + t]}{2D_j} + \frac{b_j[k + t]}{2D_j} \geq \frac{1}{2D_j} \left( \frac{1}{2D_{\text{max}}} \right)^{t} \frac{\varepsilon[k]}{2n} - g[t] \geq \frac{1}{2D_j} \frac{1}{2D_{\text{max}}} \frac{\varepsilon[k]}{2n} - g[t] > 0.
\]

[Note that in the second line we used the induction hypothesis and the fact that \( b_i[k + t] > 0 \) in the third line we used the fact that \( D_{\text{max}} > D_j > 0 \) and \( g[t] \geq 0 \), and in the fourth line we used the fact that \( g[t + 1] \geq g[t] \). Notice that the last quantity is greater than zero since we are assuming that inequality (20) holds for \( t \in \{0, 1, ..., n - 1\} \).]
we have in (10) and (11), the flow \( \Delta \) and, since stabilizes to a value \( \lim_{k \to \infty} \Delta \geq b_{l_{ij}} \frac{|l_{ij}|}{2D_l} \) or, equivalently \( \Delta f_{l_{ij}}[k + t] \geq b_{l_{ij}}[k + t] \frac{|l_{ij}|}{2D_l} > 0 \).

Since \( v_l \in T \) in the proof of Proposition 4, we can use (19) (and the fact that \( \Delta \varepsilon_i[k] \leq 0 \) for \( v_l \in T \)) to establish that

\[
\Delta \varepsilon_i[k + t] \leq |b_l[k + t + 1]| + b_l[k + t + 1] - 2 b_{l_{ij}}[k + t] \frac{|l_{ij}|}{2D_l} ,
\]

where the second inequality follows from (19) and the fact that \( \Delta f_{l_{ij}}[k] \geq b_{l_{ij}}[k + t] \frac{|l_{ij}|}{2D_l} \). [Recall that changes in the flows on edges in the first summation in (19) are nonnegative whereas changes in the flows on edges in the second summation in (19) are nonpositive.]

There are two possibilities to consider: (a) \( b_l[k + t + 1] \leq 0 \) and (b) \( b_l[k + t + 1] > 0 \), both of which lead to the desired conclusion.

(a) If \( b_l[k + t + 1] \leq 0 \), then (25) implies \( \Delta \varepsilon_i[k + t] \leq - b_{l_{ij}}[k + t] \frac{|l_{ij}|}{2D_l} \) and, since \( g[t + 1] = g[t] - \Delta \varepsilon_i[k + t] \) (with \( g[t] \geq 0 \)) we have

\[
g[t + 1] \geq g[t] + \frac{b_{l_{ij}}[k + t]}{D_l} \geq \left( \frac{1}{2D_{\max}} \right)^{t} \frac{\epsilon[k]}{2n} - \frac{1}{D_l} g[t] \geq \left( \frac{1}{2D_{\max}} \right)^{t+1} \frac{\epsilon[k]}{2n} ,
\]

which violates inequality (20) (i.e., it leads to the contradiction we are trying to establish and we are done).

(b) If \( b_l[k + t + 1] > 0 \), then (25) implies

\[
\Delta \varepsilon_i[k + t] \leq 2b_l[k + t + 1] - \frac{b_{l_{ij}}[k + t]}{D_l} .
\]

Thus, in (19), all \( \Delta \varepsilon_i[k + t] \leq 0 \), for \( v_l \in T \), we have

\[
g[t] - g[t + 1] = \Delta \varepsilon_i[k + t] \leq \Delta \varepsilon_i[k + t] \leq 2b_l[k + t + 1] - \frac{b_{l_{ij}}[k + t]}{D_l} .
\]

Using the induction invariant and the monotonicity of \( g[t] \):

\[
b_l[k + t + 1] \geq \frac{b_{l_{ij}}[k + t]}{2D_l} + \frac{1}{2} (g[t] - g[t + 1]) \geq \left( \frac{1}{2D_{\max}} \right)^{t} \frac{\epsilon[k]}{2n} - \frac{1}{2D_l} g[t] + \frac{1}{2} (g[t] - g[t + 1]) \geq \left( \frac{1}{2D_{\max}} \right)^{t+1} \frac{\epsilon[k]}{2n} - g[t + 1] .
\]

This completes the proof of Theorem 2. \( \square \)

Having completed the proof of Theorem 2, we now proceed to complete the convergence argument by proving Corollary 1.

**Proof of Corollary 1:** From Theorem 2, we have that \( \lim_{k \to \infty} \varepsilon_i[k] = \lim_{k \to \infty} \sum_{j=1}^{n} |b_{l_{ij}}[k]| = 0 \), which implies that \( \lim_{k \to \infty} b_{l_{ij}}[k] = 0 \), \( \forall v_l \in V \). From the flow updates in (10) and (11), the flow \( f_{l_{ij}}[k] \) on each edge \( (v_l, v_i) \in E \) stabilizes to a value \( f_{l_{ij}}^* \), i.e., \( f_{l_{ij}}^* = \lim_{k \to \infty} f_{l_{ij}}[k] \) exist for all edges \( (v_l, v_i) \in E \). Clearly, the algorithm described by (5)–(11) results in flows \( f_{l_{ij}}^* \) that are within the lower and upper bounds on each edge (i.e., \( l_{ij} \leq f_{l_{ij}}^* \leq u_{l_{ij}} \)). Furthermore, since \( \lim_{k \to \infty} b_{l_{ij}}[k] = 0 \), we easily obtain that \( \sum_{v_l \in N_j^-} f_{l_{ij}}^* = \sum_{v_l \in N_j^+} f_{l_{ij}} \).

**VI. DISTRIBUTED CHECKING OF CIRCULATION CONDITIONS**

Algorithm 1 presented in the previous section allows the nodes to reach a set of flows that are feasible and balanced if the circulation conditions in Theorem 1 are satisfied. In case these conditions are not satisfied, we would like the nodes to have a way to determine, in a distributed manner, that this is the case. In this section we develop a distributed algorithm (referred to as Algorithm 2) that builds on Algorithm 1 and allows the nodes to do exactly that.

We start by making the following observation. When the circulation conditions in Theorem 1 are not satisfied, Algorithm 1 converges to a set of flows that lie in the set of allowable flows as determined by the interval constraints on each edge flow, but are not necessarily balanced. The fact that the algorithm indeed converges to a set of flows is not as obvious, but can be derived from the analysis of Algorithm 1 in the previous section. To see this, suppose (for now and for simplicity) that the circulation conditions in Theorem 1 are violated for a single set of nodes \( S \), \( S \subset V \) (i.e., the conditions are not violated for any other subset of nodes). From the proof of convergence in the previous section, we see that Algorithm 1 will drive the flows on the edges that are incoming to \( S \) or outgoing from \( S \) as follows:

\[
f_{l_{ij}}[k + t] = l_{ij} , \forall (v_l, v_i) \in E_S^{-} ,
\]

\[
f_{l_{ij}}[k + t] = u_{l_{ij}} , \forall (v_l, v_i) \in E_S^{+} ,
\]

where \( E_S^{-} \) and \( E_S^{+} \) are given by (2) and (3) respectively. Nodes in the set \( S \) (the absolute balance of which will necessarily be positive) will eventually all reach positive balance and stabilize to a set of flows (and corresponding balances) that remain invariant. [We do not explicitly show this due to space limitations, but it can be shown based on the facts that (i) the flows on edges \( E_S^{-} \) \( E_S^{+} \) cannot be decreased (increased), and (ii) there are no further violations of the circulation conditions within \( S \).] Moreover, nodes in the set \( V - S \) (the absolute balance of which will necessarily be negative) will all reach negative or zero balance and stop updating the flows. The situation is slightly more complicated if there are multiple/intersecting sets for which the circulation conditions are violated, but the important fact for the analysis in this section is that the flows in Algorithm 1 converge even when the conditions in Theorem 1 are violated (this is also seen in the next section when we present simulations for various types of graphs, including graphs in which the circulation conditions are violated — see Fig. 4 which shows how flows are updated on a graph of 20 nodes in which the circulation conditions are violated).

Clearly, once the nodes converge to a set of flows \( f_{l_{ij}}^* \), \( \forall (v_l, v_i) \in E \), a simple way to determine that the resulting solution does not correspond to a set of balanced flows is...
to run an average consensus on the absolute values of the resulting balances $b^*_j$, $\forall v_j \in V$. The graph is balanced (i.e., $b_j = 0$, $\forall v_j \in V$) if and only if the average of the absolute balances satisfies
\[
\frac{\sum_{v_i \in V} |b_i|}{n} = 0.
\]

If each node $v_j$ was aware of its value $b^*_j$, since the communication graph $G_u$ is undirected, one simple way to obtain the above average is to run a linear iterative scheme where each node maintains a variable $x_j[k]$ (initialized at $x_j[0] = |b^*_j|$) and, at each iteration $k$, each node updates its value as a weighted sum of its own value and the values of its neighbors $[6], [22]$. In particular,
\[
x_j[k + 1] = p_{jj}x_j[k] + \sum_{i \in N_j} p_{ji}x_i[k],
\]
where $N_j = N^+_j \cup N^-_j$ is the set of neighbors of node $v_j$ and $p_{ji}$ are a set of (fixed) weights, chosen so that the weight matrix $P = [p_{ji}]$ (with weights $p_{ji}$ satisfying $p_{ji} = 0$ if $i \notin N_j \cup \{j\}$) is doubly stochastic with a simple eigenvalue at 1 and all other eigenvalues having magnitude smaller than 1. In such case, it can be shown that iteration (26) asymptotically reaches average consensus [13], i.e.,
\[
\lim_{k \to \infty} x_j[k] = \frac{\sum_{v_i \in V} x_i[0]}{n}, \forall v_j \in V.
\]

Given an undirected graph $G_u = (V, E_u)$, there are many different ways of having the nodes distributively assign weights $p_{ji}$ such that the resulting matrix $P = [p_{ji}]$ is a primitive doubly stochastic matrix. For instance, assuming the nodes know the total number of nodes $n$ or an upper bound $n' \geq n$, each node $v_j$ can choose
\[
p_{ji} = \begin{cases} 
\frac{1}{n'}, & \text{if } (v_j, v_i) \in E_u, \\
0, & \text{if } (v_j, v_i) \notin E_u, \\
1 - \frac{|N_j|}{n'}, & \text{if } v_j = v_i.
\end{cases}
\]

In practice, the value of $b^*_j$ only becomes available to node $v_j$ asymptotically. Thus, we implement a running average consensus algorithm where we also update the balance of node $v_j$. In particular, in parallel to Algorithm 1 (which available to each node $v_j$ the value $b_j[k + 1]$) we run the iteration:
\[
x_j[k + 1] = p_{jj}x_j[k] + \sum_{i \in N_j} p_{ji}x_i[k] + |b_j[k + 1]| - |b_j[k]|, \Delta b_{abs}[k]
\]
with initial conditions $x_j[0] = 0$ and $\Delta b_{abs}[0] = 0$. The above iteration resembles iteration (26) and ensures that $\sum_{v_i \in V} x_i[k] = \sum_{v_i \in V} |b_i[k]|$ for all $k$ (because the matrix $P = [p_{ji}]$ is column stochastic). Eventually, for large $k$, since the flows (and thus the balances) converge, $\Delta b_{j}[k]$ converges to zero and the above iteration becomes equivalent to iteration (26). Thus, the nodes reach consensus on the average of the absolute values of their eventual balances, i.e.,
\[
\lim_{k \to \infty} x_j[k] = \frac{\sum_{v_i \in V} |b_i^*|}{n}, \forall v_j \in V.
\]

Algorithm 2: Distributed algorithm enhanced for detection of violation of circulation conditions

Each node $v_j \in V$ separately does the following:

Input: $l_{ij}, u_{ij}, \forall v_i \in N^-_j$
Input: $l_{ij}, u_{ij}, \forall v_i \in N^+_j$
Input: $n'$ upper bound on number of nodes ($n' \geq n$)
Output: $\tilde{f}_{ij}, \forall v_i \in N^-_j$
Output: $\tilde{f}_{ij}, \forall v_i \in N^+_j$
Output: Maintain $x_j[k]$ as indicator for infeasible circulation conditions

begin
Set $f_{ij}[0] = \frac{l_{ij} + u_{ij}}{2}, \forall v_i \in N^-_j$
Set $f_{ij}[0] = \frac{l_{ij} + u_{ij}}{2}, \forall v_i \in N^+_j$
Set $D_j = D^-_j + D^+_j$
Set $p_{ij} = \frac{1}{n'}$ for $j \in N_j, p_{ij} = 1 - \frac{|N_j|}{n'}$
Set $x_j[0] = 0, \tilde{b}_j[0] = 0, and x_i[0] = 0, \forall v_i \in N_j$

foreach iteration, $k = 0, 1, ..., do
Calculate:
\[b_j[k] = \sum_{v_i \in N^-_j} f_{ij}[k] - \sum_{v_i \in N^+_j} f_{ij}[k]\]
\[x_j[k + 1] = p_{jj}x_j[k] + \sum_{i \in N_j} p_{ji}x_i[k] + |b_j[k]| - |\tilde{b}_j[k]|\]
Set: $\tilde{b}_j[k] = b_j[k]$
Set: $\tilde{b}_j[k] = \begin{cases} b_j[k], & \text{if } b_j[k] > 0, \\
0, & \text{otherwise.}
\end{cases}$
Transmit: $\tilde{b}_j[k]$ and $x_j[k + 1]$ to $v_i \in N^-_j$ and $v_i \in N^+_j$

Receive: $\tilde{b}_l[k]$ from all $l \in N^-_j$
Receive: $\tilde{b}_l[k]$ from all $l \in N^+_j$

Calculate:
\[f_{ij}[k + 1] = f_{ij}[k] + \frac{1}{2} \left( \tilde{b}_l[k] - \tilde{b}_l[k] \right), v_i \in N^-_j\]
\[f_{ij}[k + 1] = f_{ij}[k] + \frac{1}{2} \left( \tilde{b}_l[k] - \tilde{b}_l[k] \right), v_i \in N^+_j\]
Set:
\[f_{ij}[k + 1] = \begin{cases} f_{ij}[k + 1], & \text{if } l_{ij} \leq f_{ij}[k + 1] \leq u_{ij}, \\
l_{ij}, & \text{if } f_{ij}[k + 1] > u_{ij} \\
l_{ij}, & \text{if } f_{ij}[k + 1] < l_{ij}
\end{cases}
\]
\[f_{ij}[k + 1] = \begin{cases} f_{ij}[k + 1], & \text{if } l_{ij} \leq f_{ij}[k + 1] \leq u_{ij}, \\
l_{ij}, & \text{if } f_{ij}[k + 1] > u_{ij} \\
l_{ij}, & \text{if } f_{ij}[k + 1] < l_{ij}
\end{cases}
\]

end
This average is greater than zero if and only if the circulation conditions are violated, providing a distributed way for the nodes to determine the violation. The algorithm pseudocode is provided in Algorithm 2.

**Example 2:** In this example, we revisit the digraph in Example 1 where we modify (decrease) the upper bounds on two edges, so as to create a violation of the circulation conditions. In particular, we take \( u_{1,7} = 2 \) and \( u_{2,7} = 4 \), and run Algorithm 2. On the left of Fig. 3, we plot the values of the flows \( f_{j|i}[k] \) for each \((v_j, v_i) \in E\). It can be observed that the flows eventually stabilize to fixed values as argued in this section. In the middle of Fig. 3, we plot the evolution of the absolute (total) imbalance \( \varepsilon[k] \) against the iteration \( k \). Notice that \( \varepsilon[k] \) is monotonically non-increasing, but does not go to zero. On the right of Fig. 3, we plot the values \( x_j[k] \) in Algorithm 2 for each node \( v_j \in V \). These values eventually converge to the positive value 2.2857 which is the average of the eventual absolute balance (which is 16 in this case). Note that the running averages of the absolute balances do not evolve monotonically.

### VII. Simulation Results

In this section, we present simulation results for Algorithm 1. Specifically, we first present numerical results for a random graph of size \( n = 20 \) illustrating the behavior of Algorithm 1 for two different cases: (i) the case when the circulation conditions in Theorem 1 do not hold, thus, a set of flows that balance the digraph cannot be obtained; (ii) the case when the circulation conditions in Theorem 1 hold and a set of flows that balance the graph can be obtained. We also present numerical results for graphs of various sizes (\( n = 20, 50, \) and 100) when the circulation conditions in Theorem 1 hold and a set of flows that balance the graph can be obtained. All graphs are randomly generated by creating, independently for each ordered pair \((v_j, v_i)\) of two nodes \( v_j \) and \( v_i \) \((v_j \neq v_i)\), a directed edge from node \( v_i \) to node \( v_j \) with probability \( p \) \((0 < p < 1)\) while the flows are initialized at the middle of the feasible interval i.e., \( f_{ji}[0] = (l_{ji} + u_{ji})/2 \).

Fig. 4 shows what happens in the case of a randomly created graph of 20 nodes, in which the circulation conditions in Theorem 1 do not hold. In the first case, we plot the absolute (total) imbalance \( \varepsilon[k] = \sum_{j=1}^{n} |b_j[k]| \), and in the second case, the node balances \( b_j[k] \) as a function of the number of iterations \( k \) for the distributed algorithm. The plots suggest that the proposed distributed algorithm (Algorithm 1) is unable to obtain a set of flows that balance the corresponding digraph. Nevertheless, as argued in Section VI, the flows converge to a feasible set of values (yet not balanced).

Fig. 5 shows the same case as Fig. 4, with the difference that the circulation conditions in Theorem 1 hold. Here, the plots suggest that Algorithm 1 is able to obtain a set of flows that balance the corresponding digraph.

Fig. 6 shows what happens in the case of 100 graphs of 20 (left), 50 (middle), and 100 (right) nodes each when the circulation conditions in Theorem 1 hold. We plot the average absolute (total) imbalance \( \varepsilon[k] = \sum_{j=1}^{n} |b_j[k]| \) as a function of the number of iterations \( k \) for Algorithm 1. The plot suggests that Algorithm 1 is able to obtain a set of flows that balance the corresponding graph.

### VIII. Concluding Remarks

In this paper, we introduced and analyzed a distributed algorithm for assigning flows to the edges of a commodity network, described by a digraph, so as to balance the in- and out-flows on each node, while satisfying capacity limits on the edges. In addition, we provided an enhancement to such algorithm that allows the nodes to determine whether or not a flow assignment exists.
In the future, we plan to investigate ways of relaxing the assumption regarding bi-directional communication between neighboring nodes. Since in many applications the communication topology does not necessarily match the physical one, we plan to investigate ways to enhance the algorithm proposed here to allow for different communication topologies.

REFERENCES


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