Marking observer of labeled Petri nets with uncertainty in the initial marking

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Abstract—In this paper we consider marking estimation in labeled Petri nets whose initial marking is known to belong to a given convex set. We allow for silent transitions (i.e., transitions labeled with the empty word) and indistinguishable transitions (i.e., transitions sharing the same label with other transitions). We demonstrate that all sets of markings consistent with a given sequence of observations can be described in linear algebraic terms (as a union of convex sets) and a marking observer may be constructed offline under appropriate boundedness assumptions. This reduces the problem of computing the set of markings consistent with a given observation sequence to the problem of moving along a path in a labeled directed graph.

Index Terms—Petri nets, discrete event systems, state estimation.

I. INTRODUCTION

In this paper we deal with the problem of designing a state marking observer for a given Petri net. This problem has been extensively investigated in the literature in the last decades [2], [3], [6], [7], [9], and a good survey on state estimation in Petri nets can be found in [5].

In [6] the observability problem has been addressed under the assumptions that (i) the initial state of the Petri net is completely unknown, or at most known to belong to a given convex set, and (ii) all transitions are observable. Later on, in [2], [3] a different problem statement has been considered: the initial marking is assumed to be known but only a subset of transitions is assumed to be observable. In particular, some of them may be silent and others may be indistinguishable, i.e., their firing generates the same observation. The above two forms of nondeterminism are easily modeled in terms of labeled Petri nets: silent transitions are labeled with the empty word $\varepsilon$, and indistinguishable transitions are labeled with the same symbol in a given alphabet $E$.

In this paper we simultaneously assume uncertainty in the initial marking and transitions that may be silent and/or indistinguishable. In particular, we assume that the initial marking is only known to belong to a given convex set. To the best of our knowledge this is the first time where these two forms of nondeterminism are considered simultaneously.

We first prove that the set of markings that can be reached following the firing of any sequence of transitions, starting from a given initial convex set of markings, is convex as well. Moreover, the number of such convex sets is finite if the net system is bounded for all initial markings in the given convex set. This enables us to define an extended version of the reachability graph that we call Extended Reachability Graph (ERG). The initial node of ERG is associated with the convex set describing the set of possible initial markings. Then, each different node of the ERG is associated with a convex set of markings that describes the set of markings consistent with a given sequence of transitions. Edges are labeled with transitions, thus the convex set associated with a node that can be reached following a path labeled with a given sequence of transitions describes the set of markings that can be reached from the set of possible initial markings following the firing of such a sequence of transitions.

Starting from the ERG, an observer can be easily computed by merging all nodes that are connected by edges labeled with silent transitions and all nodes that exit from the same node and whose output edges are transitions sharing the same label. Obviously, the resulting sets associated with the nodes of the observer generally form a non convex set, but they can be described as unions of convex sets. Specifically, they can still be described in linear algebraic terms using the quite standard approaches in [1] based on the big-M technique. This approach allows us to treat the case of a set of possible initial markings in a labeled Petri net with complexity similar to the complexity of the case where the initial marking is known exactly.

II. BACKGROUND ON LABELED PETRI NETS

In this section we recall the formalisms used in the paper. For more details on PNs we refer to [8].

A Place/Transition net (P/T net) is a structure $N = (P,T, \text{Pre}, \text{Post})$, where $P$ is a set of $m$ places; $T$ is a set of $n$ transitions; $\text{Pre} : P \times T \rightarrow \mathbb{N}$ and $\text{Post} : P \times T \rightarrow \mathbb{N}$ are the pre– and post– incidence functions that specify the nonnegative arc weights; $C = \text{Post} – \text{Pre}$ is the incidence matrix.

A marking is a vector $M : P \rightarrow \mathbb{N}$ that assigns to each place of a P/T net a nonnegative integer number of tokens, represented by black dots. We denote by $M(p)$ the marking of
place). A P/T system or net system $(N, M_0)$ is a net $N$ with an initial marking $M_0$. A transition $t$ is enabled at $M$ iff $M \geq Pre(t) \cdot M$. If $M$ is an initial marking, we write $M(\sigma)$ to denote that the sequence of transitions $\sigma = t_j \ldots t_k$ is enabled at $M$, and we write $M[\sigma]M'$ to denote that the firing of $\sigma$ yields $M'$. We also write $t \in \sigma$ to denote that a transition $t$ is contained in $\sigma$. The set of all sequences that are enabled at the initial marking $M_0$ is denoted by $L(N, M_0)$, i.e., $L(N, M_0) = \{ \sigma \in T^* \mid M(\sigma) \}$.

Given a sequence $\sigma \in T^*$, we call $\pi : T^* \rightarrow N^n$ the function that associates with $\sigma$ a vector $\bar{y} \in N^n$, named the firing vector of $\sigma$. In particular, $\bar{y} = \pi(\sigma)$ is such that $\bar{y}(t) = k$ if the transition $t$ is contained $k$ times in $\sigma$.

A marking $M$ is reachable in $(N, M_0)$ iff there exists a firing sequence $\sigma$ such that $M_0[\sigma]M$. The set of all markings reachable from $M_0$ defines the reachability set of $(N, M_0)$ and is denoted by $R(N, M_0)$. A net system $(N, M_0)$ is bounded if there exists a positive constant $k$ such that, for $M \in R(N, M_0)$, $M(p) \leq k$.

A labeling function $L : T \rightarrow E \cup \{ \varepsilon \}$ assigns to each transition $t \in T$ either a symbol from a given alphabet $E$ or the empty string $\varepsilon$. We denote by $T_0$ the set of transitions labeled by the empty word $\varepsilon$. With a slight abuse of notation we also extend the definition of $L$ to sequences: for $\sigma \in T^*$, $L(\sigma t) = L(\sigma) L(t)$ where $L(t)$ is the label assigned to transition $t$, that may either be a symbol in $E$ or the empty string, and $L(\varepsilon) = \varepsilon$.

III. CONVEXITY PROPERTIES

In this section we first prove that starting from a given convex set of initial markings, the set of markings that can be reached following the firing of a sequence of transitions is convex as well. Second, we show that, if the net system is bounded for all initial markings in the starting convex set, then the number of different convex sets that can be obtained following a possible firing sequence (i.e., a firing sequence enabled at the initial set), is finite as well.

Let
\[
\mathcal{M} = \{ M \in N^m \mid AM \leq \bar{b}, M \leq \bar{b}_u, M \geq \bar{b}_l \}
\]  
(1)
be a given convex set of markings, where matrix $A \in Z_{n_e \times m}$ is such that each row contains at least two non-zero entries, i.e., it describes a constraint that involves at least two places; $n_e$ is the number of such constraints and we assume that none of them is redundant. The other two inequalities assign, respectively, an upper and a lower bound to places, that may also be equal to zero. Obviously, we have $\bar{b}_u, \bar{b}_l \in N^m$ and $\bar{b}_u \geq \bar{b}_l$.

**Proposition 1:** Let $M$ be a convex set of markings defined as in equation (1). The set of markings $\mathcal{R}(M, t)$ reached by firing any transition $t \in T$ starting from any marking in $M$ is convex and is equal to
\[
\mathcal{R}(M, t) = \{ M \in N^m \mid AM \leq \bar{b} + AC(\cdot ,t), M \leq \bar{b}_u + C(\cdot ,t), M \geq \max\{ \bar{b}_l, Pre(\cdot , t) \} + C(\cdot , t) \},
\]  
(2)
where the maximum operator is taken componentwise.

Proof: We first observe that the set of markings in $M$ that enable $t \in T$ is convex. In particular, it is equal to
\[
\mathcal{M}_{en}(t) = \{ M \in N^m \mid AM \leq \bar{b}, M \leq \bar{b}_u, M \geq \max\{ \bar{b}_l, Pre(\cdot , t) \} \}.
\]  
(3)
Now, since by the state equation the firing of $t$ at a given marking $M$ leads to a marking $M' = M + C(\cdot , t)$, replacing $M$ with $M' - C(\cdot , t)$ in (3) leads to equation (2).

The previous result can be easily generalized to the case of a sequence of transitions $\sigma \in T^*$.

**Proposition 2:** Let $M$ be a convex set of markings defined as in equation (1). The set of markings $\mathcal{R}(M, \sigma)$ reached by firing any transition sequence $\sigma = t_{i_1} t_{i_2} \ldots t_{i_k} \in T^*$ starting from any marking in $M$ is convex and can be computed via the following steps:

1. Let $\bar{b}_{i_0} = \bar{b}_i$.
2. For all $j = 1, \ldots , k$,
   a. let $\bar{b}_{i,j} = \max\{ \bar{b}_{i,j-1}, Pre(\cdot , t_{i_j}) \} + C(\cdot , t_j)$.
3. Let $\bar{b}_{i,\sigma} = \bar{b}_{i,k}$.
4. Let
   \[
   \mathcal{R}(M, \sigma) = \{ M \in N^m \mid AM \leq \bar{b} + AC\bar{\sigma}, M \leq \bar{b}_u + C\bar{\sigma}, M \geq \bar{b}_l \}.
   \]  
(4)

Proof: It follows from recursively applying Proposition 1.

The above results have two important practical implications.

- The number of constraints ($n_e$) necessary to describe the set of consistent markings in which the system may be, starting from any set in the form (1) and firing any sequence $\sigma \in T^*$, is constant.
- The structure of such a set of constraints does not depend on $\sigma$. Indeed, only the right-hand-side vector varies with the considered sequence.

As a result of the above discussion, the following proposition can be proved.

**Proposition 3:** Consider a Petri net $N$ whose initial marking belongs to a given convex set $\mathcal{M}_0$ defined as in equation (1). Let
\[
\mathcal{R}(\mathcal{M}_0) = \{ \mathcal{R}(\mathcal{M}_0, \sigma) \in 2^{N^m} \mid \sigma \in L(N, M_0), M_0 \in \mathcal{M}_0 \}
\]  
(5)
be the set of admissible sets of markings that $N$ can reach when all transition sequences firable at any $M_0 \in \mathcal{M}_0$ are considered. If $(N, M_0)$ is bounded for any initial marking $M_0 \in \mathcal{M}_0$, then the cardinality of $\mathcal{R}(\mathcal{M}_0)$ is finite.

Proof: Let us preliminary observe that the vector $C\bar{\sigma}$, where $\sigma \in L(N, M_0)$ and $M_0 \in \mathcal{M}_0$, may only take a finite number of values if the net is bounded for any initial marking in $\mathcal{M}_0$. Indeed, two different cases may occur.

- For all $M_0 \in \mathcal{M}_0$ the language $L(N, M_0)$ is finite. In such a case only a finite number of firing vectors $\bar{\sigma}$ may occur (since only a finite number of sequences $\sigma$
are possible), thus $C\sigma$ may obviously take only a finite number of values.

- For some $M_0 \in \mathcal{M}_0$ the language $L(N, M_0)$ is not finite. Languages that are not finite surely contain repetitive sequences. In general, repetitive sequences may either be increasing or stationary. In the first case it is $C\sigma \geq 0$; in the second case it is $C\sigma = 0$. However, increasing sequences make the marking of some places grow indefinitely, thus they cannot occur in bounded systems, as in the case at hand. This means that, if the language is not finite, the only repetitive sequences are stationary. Therefore, as in the case of finite languages, in this case of infinite languages, the vector $C\sigma$ may only take a finite number of values starting from any marking $M_0$ in $\mathcal{M}_0$. Summarizing, only a finite number of right-hand side vectors in the first two inequalities defining the generic set $\mathcal{R}(M_0, \sigma)$ can be obtained. This proves the statement since by the second inequality in (4) the marking of no place may grow indefinitely regardless of the values assumed by $b_{l,\sigma}$ in the third inequality.

\[ \square \]

IV. EXTENDED REACHABILITY GRAPH

In this section we define a particular directed and labeled graph, called Extended Reachability Graph (ERG) where a different node is associated with each element in $\mathcal{R}$. Arcs are labeled with transitions. An arc labeled $t$ goes from one node to another node if and only if transition $t$ may fire at some marking in the first node and its firing leads to the set of markings in the second node. The following algorithm summarizes the main steps for the construction of the ERG.

\textbf{Algorithm 4: Extended Reachability Graph.}

\begin{enumerate}
  \item The initial node of the graph is the set of admissible initial markings in a convex set $\mathcal{M}_0$ (as described in (1)). This node is initially unmarked.
  \item While there exists an unmarked node $\mathcal{M}$ of the graph, do:
    \begin{enumerate}
      \item For each transition $t$ enabled at some marking in $\mathcal{M}$, i.e., such that $\mathcal{M}_t(t)$ is not empty:
        \begin{enumerate}
          \item Compute the convex set of markings $\mathcal{M}' = \mathcal{R}(\mathcal{M}, t)$ according to equation (2).
          \item If no node $\mathcal{M}'$ is on the graph, add a new node $\mathcal{M}'$ to the graph.
          \item Add an arc $t$ from node $\mathcal{M}$ to node $\mathcal{M}'$.
        \end{enumerate}
      \item Mark node $\mathcal{M}$ “marked”.
    \end{enumerate}
  \item Remove all labels.
\end{enumerate}

\textbf{Example 5:} Consider the Petri net in Fig. 1 whose initial marking is known to belong to the set

\[ \mathcal{M}_0 = \{ M \in \mathbb{R}^m \mid M(p_1) + M(p_2) \leq 4, \]
\[ M \geq [0 \ 0 \ 0 \ 0 \ 0]^T, \]
\[ M \leq [4 \ 4 \ 0 \ 0 \ 0]^T \}. \] (6)

According to the notation in eq. (1), we have $A = [1 \ 1 \ 0 \ 0 \ 0]^T$, $b = 4$, $b_1 = [0 \ 0 \ 0 \ 0 \ 0]^T$, and $b_u = [4 \ 4 \ 0 \ 0 \ 0]^T$.

Based on Algorithm 4 we obtain the ERG in Fig. 2 containing 28 different states. Table I summarizes the characteristic values of all such states, namely $b$, $b_1$ and $b_u$.

It is mandatory to observe that the values of the parameters in Table I are not exactly the same as those computed at Step 2.i of Algorithm 4. Consider as an example $\mathcal{R}(\mathcal{M}_0, t_1)$. Based on Step 2.i it holds that $b_u = [3 \ 3 \ 1 \ 1 \ 0]^T$ rather than $b_u = [2 \ 2 \ 1 \ 1 \ 0]^T$ as reported in Table I. However, it is easy to observe that, since $b = 2$, the two different values of the upper bounds provide the same set of feasible markings. Clearly, avoiding redundant constraints is a very important issue since it allows to immediately establish if a node already exists in the graph or not (see Step 2.ii of Algorithm 4). In this numerical example eliminating redundant constraints has been trivial. In more complex examples, systematic reduction techniques are obviously required. For the sake of brevity, this issue is not addressed in this conference paper. 

\[ \square \]
V. Marking observer

In this section we deal with the problem of computing the set of markings in which the system may be, given the set of possible initial markings and a sequence of observations \( w \in E^* \). More precisely, if \( M_0 \) is the convex set of markings in which the system is known to initially be, and \( w \in E^* \) is the observed sequence of labels, we want to compute the set of markings consistent with the observation \( w \), namely

\[
C(M_0, w) = \{ M \in \mathbb{N}^m \mid M_0(\sigma)M, \; M_0 \in M_0, \; \mathcal{L}(\sigma) = w \}. \tag{7}
\]

To accomplish this systematically, moving online most of the calculations, we define a Marking Observer (MO). The MO is a deterministic graph whose nodes contain sets of nodes of the ERG and whose arcs are labeled with symbols from the alphabet \( E \). Given the MO, the problem of computing the set of markings consistent with a given sequence of observations simply reduces to the problem of following a labeled path in a deterministic graph: the set of consistent markings is then given by the union of convex sets associated with subsets of ERG nodes captured by the MO node reached.

The approach we propose to compute the MO is based on an important result that enables us to describe in linear algebraic terms the union of a finite number of convex sets, which is obviously in general a non convex set. Such a result is known as the big-M technique and was first proposed by Bemporad et al. in [1].

**Proposition 6:** Assume that we have \( k \) sets of constraints of the form \( A_1M \leq \vec{b}_1, \ldots, A_kM \leq \vec{b}_k \). The union of such sets can be described as follows, by simply introducing \( k \) binary variables \( z_1, z_2, \ldots, z_k \):

\[
\begin{align*}
A_1M - \vec{b}_1 & \leq Kz_1, \\
\vdots \\
A_kM - \vec{b}_k & \leq Kz_k, \\
z_1 + \ldots + z_k & = k - 1, \\
z_1, \ldots, z_k & \in \{0, 1\},
\end{align*}
\]

where \( K \) is a very large constant.

The above description follows from the fact that, since the sum of all \( z_i \)'s is equal to \( k - 1 \), then only one \( z_i \) at a time is null, while all the others are equal to one. As an example, if \( z_1 = 0 \), it means that only the first constraint is active while all the others are redundant.

We now present an algorithm that summarizes the main steps for the construction of the MO. Basically the main steps are the same as those required to compute a deterministic finite automaton (the MO) equivalent to a given nondeterministic one (the ERG) [4]. The only significant difference is that, in each node, instead of having a set of states, we have the union of a set of linear algebraic constraints.

In the following, for simplicity of presentation, we denote the ERG as a non deterministic automaton \( G' = (X, E, \delta', x_0') \), where \( X \) is the set of states (a different state is associated with each convex subset of \( \mathcal{R}(M_0) \) of the form in (4)); \( E \) is the alphabet associated with the considered labeled Petri net; \( \Delta \subseteq X \times (E \cup \{\varepsilon\}) \times X \) is the transition relation.

The MO that we want to obtain is a deterministic automaton \( G = (X', E, \delta, x_0) \) equivalent to \( G' \), namely such that their languages coincide. More specifically, \( G' \) results from the determinization of \( G \), taking also into account the unobservable reach of each state: \( X' = 2^X \), \( x_0' \) is defined below and \( \delta' \) is the transition function \( \delta': X' \times E \rightarrow X' \), rather than the transition relation. Each state in \( X' \) is associated with a set of linear algebraic constraints defined as in the following algorithm.

**Algorithm 7: Marking Observer.**

1. For all states \( x \in X \) compute

\[
D(x) = \{ \bar{x} \in X \mid (x, \varepsilon, \bar{x}) \in \Delta^* \},
\]

i.e., the set of the states that can be reached from \( x \) by executing zero or more \( \varepsilon \) transitions. By definition, we have \( x \in D(x) \).

2. For all states \( x \in X \) and all symbols \( e \in E \), compute

\[
D_e(x) = \{ \bar{x} \in X \mid (x, e, \bar{x}) \in \Delta \},
\]

i.e., the set of the states reachable from \( x \) executing \( e \)-transitions.

3. Let \( x_0' = D(x_0) \), i.e., the initial state of \( G' \) is given by the set of states \( G \) reachable from \( x_0 \) executing zero or more \( \varepsilon \)-transitions.

4. Let \( X' = \emptyset \) and \( X'_{\text{new}} = \{ x_0' \} \).

At the end of the algorithm \( X' \subseteq 2^X \) is the set of states of \( G' \), while the set \( X'_{\text{new}} \) contains, at each iteration, the set of states of \( G' \) that have yet to be explored.
5. Consider a state \( x' \in X'_{\text{new}} \).
   a. For all \( e \in E \):
      i. Define the sets \( A(x', e) = \bigcup_{x \in A(x', e)} D_e(x) \) and
      \[
      B(x', e) = \bigcup_{x \in A(x', e)} D(x).
      \]
      The first set represents the states of \( \mathcal{G} \) reachable from any state in \( x' \)
      executing an \( e \)-transition. The second set represents the set of states of \( \mathcal{G} \)
      reachable from any state in \( A(x', e) \) executing zero or more \( e \)-transitions.
   ii. Consider \( \bar{x}' = B(x', e) \) as the state of \( \mathcal{G}' \) reached
      from \( x' \) executing \( e \) and thus let \( \delta'(x', e) = \bar{x}' \).
   iii. If \( \bar{x}' \notin X' \cup X'_{\text{new}} \) let \( X'_{\text{new}} = X'_{\text{new}} \cup \{ \bar{x}' \} \).
      Let \( X' = X' \cup \{ x' \} \) and \( X'_{\text{new}} = X'_{\text{new}} \setminus \{ x' \} \).
   6. If \( X'_{\text{new}} \neq \emptyset \) goto Step 5.
7. For all \( x' \in X' \):
   a. Let \( \tilde{X} \subseteq X \) be the set of states of \( \mathcal{G} \) belonging to
      \( x' \). Each state \( x \in \tilde{X} \) corresponds to a different set
      \( \mathcal{R}(\mathcal{M}, \sigma) \in \mathcal{R}^{(0)}(\mathcal{M}) \).
   b. Let \( k = |X| \).
   c. Let \( \sigma_j \) be the sequence of transitions that leads from
      \( M_0 \) to the generic \( j \)-th state of \( \tilde{X} \) (\( j = 1, \ldots, k \)).
   d. Let us rewrite \( \mathcal{R}(\mathcal{M}_0, \sigma_j) \) as:
      \[
      \mathcal{R}(\mathcal{M}_0, \sigma_j) = \{ M \in \mathbb{N}^m \mid \bar{A}M \leq \bar{b}_j \}
      \]
      where according to Proposition 2 it is
      \[
      \bar{A} = \begin{bmatrix}
      A & 0_{n \times m} & 0_{n \times m} \\
      0_{m \times m} & I_m & 0_{m \times m} \\
      0_{m \times m} & 0_{m \times m} & -I_m
      \end{bmatrix},
      \]
      and
      \[
      \bar{b}_j = \begin{bmatrix}
      \bar{b} + AC\sigma_j \\
      \bar{b}_u + C\bar{\sigma}_j \\
      -\bar{b}_a, \sigma_j \end{bmatrix},
      \]
      where \( I_m \) is the identity matrix, and \( 0_{n \times m} \) and
      \( 0_{m \times m} \) are null matrices of dimensions \( n \times m \) and
      \( m \times m \), respectively.
   e. Associate with \( x' \) the following set of constraints:
      \[
      \begin{align*}
      \bar{A}M - \bar{b}_1 & \leq Kz_2 \\
      \vdots \\
      \bar{A}M - \bar{b}_k & \leq Kz_k \\
      z_1 + \ldots + z_k & = k - 1 \\
      z_1, \ldots, z_k & \in \{0, 1\} \\
      M & \in \mathbb{N}^m, \text{ for some very large constant } K.
      \end{align*}
      \]
   Example 8: Consider again the labeled Petri net in Fig. 1
   whose ERG is reported in Fig. 2 and illustrated in Example 5.
   Replacing transitions with their labels in Fig. 2 we obtain the
   non deterministic automaton in Fig. 3.

   Using Algorithm 7 we compute the MO in Fig. 4 consisting
   of 39 states defined as detailed in Table II. As an example,
   consider state \( x'_7 \) that can be reached from an initial marking
   after observing the sequence \( ab \); according to Step 7.e of

   Algorithm 7, we obtain that the set of consistent markings
   given the observation \( ab \) is:
   \[
   \begin{align*}
   \bar{A}M - \bar{b}_1 & \leq Kz_1 \\
   \bar{A}M - \bar{b}_2 & \leq Kz_2 \\
   z_1 + z_2 & = 1, \\
   z_1, z_2 & \in \{0, 1\}.
   \end{align*}
   \]
   where
   \[
   \bar{A} = \begin{bmatrix}
   A & 0_{1 \times 5} & 0_{1 \times 5} \\
   0_{5 \times 5} & I_5 & 0_{5 \times 5} \\
   0_{5 \times 5} & 0_{5 \times 5} & -I_5
   \end{bmatrix},
   \]
   \[
   \bar{b}_1 = [2 \; 0 \; 0 \; 0 \; 0 \; 1 \; 0 \; 1 \; 0 \; 1]^T
   \]
   and
   \[
   \bar{b}_2 = [2 \; 0 \; 0 \; 1 \; 1 \; 0 \; 0 \; 0 \; 1 \; 1]^T.
   \]
   Note that the values of \( \bar{b}_1 \) and \( \bar{b}_2 \) can be easily deduced looking
   at the definition of states \( x_3 \) and \( x_4 \) in Table I.

   This implies that there are 6 markings consistent with the
   firing of \( ab \), namely \( M'_1 = [1 \; 1 \; 0 \; 1 \; 1]^T \), \( M'_2 = [2 \; 0 \; 1 \; 0 \; 0 \; 1 \; 1]^T \),
   \( M'_3 = [0 \; 2 \; 0 \; 1 \; 1]^T \), \( M'_4 = [1 \; 1 \; 1 \; 0 \; 1 \; 1]^T \), \( M'_5 = [2 \; 0 \; 1 \; 0 \; 1 \; 1]^T \),
   \( M'_6 = [0 \; 2 \; 1 \; 1 \; 0 \; 1]^T \).

   VI. CONCLUSIONS AND FUTURE WORK

   This paper addressed the problem of state estimation of
   labeled Petri nets with partial observation and uncertainty in
   the initial marking. Assuming that the initial marking of
   the considered labeled Petri net system belongs to a given convex
set, we first proved that the set of markings that can be reached by firing a sequence of transitions, starting from the given initial convex set of markings, is convex as well, and under certain assumptions, it is also bounded. Then, we presented the Extended Reachability Graph, that is an extended version of the reachability graph. Finally, starting from such a graph we illustrated an algorithm for the construction of a deterministic graph called Modified Observer.

In future work we plan to apply the proposed procedure to state feedback control and fault diagnosis. Moreover, we plan to analyze both current-state opacity and initial-state uncertainty in labeled Petri net systems with partial observation and uncertainty in the initial marking.

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REFERENCES